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0-104 On the Growth Order of Solutions of the Schrödinger equation

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Abstract

The object of this article is to extend J. Uchiyama's result on the growth order of "eigenfunctions" corresponding to "positive eigenvalues" of the Schrödinger operator $-\triangle+V(x)$. The potential function $V(x)$ is assumed to satisfy the differential inequality $rV_r+2\beta V\leqq 0$, $0<\beta<1$ ($r=|x|$), and the region $\Omega\subset R^n$ where the equation is considered is assumed only to include the outside of some sphere. The proof is simple by adopting S. Agmon's argument on a differential equation in a Hilbert space.

§ 1. Assumptions and theorem

The absence of positive eigevalues of the Schrödinger operator with singular potential was studied by several authors ([1], [2], [6], [7]), and J. Uchiyama [5] gave the most comprehensive results. (His results are quoted later.) The framework of problem and the assumptions are the same as of Uchiyama's. Nemely, we consider non-zero solutions of the equation

$$-\triangle u+V(x)u=\lambda u \qquad \text{in } \Omega, \tag{1.1}$$

(Ω is a region in R^n, λ is a positive constant and $\triangle$ denotes the Laplacian operator) with the following assumptions:

Assumption 1. *Ω includes the set E_{r_0}, the exterior of a sphere with the radius r_0.*

Assumption 2. *$V(x)$ is a real-valued function possessing the radial derivative $V_r(x)=\dfrac{\partial}{\partial r}V(x)$, $r=|x|$.*

Assumption 3. *$V(x)$ satisfies "Stummel's condition":*

$$\int_{|x-y|\leqq 1}\frac{|V(y)|^2}{|x-y|^{n-4+\alpha}}dy\leqq M$$

with some constants α and M independent of x. (If $n+\alpha\leqq 4$ we assume $\int_{|x-y|\leqq 1}|V(y)|^2dy\leqq M$ instead.)

Assumption 4. *$V(x)$ and $V_r(x)$ satisfy the inequality*

$$rV_r(x)\leqq -2\beta V(x)$$

for almost every x with some constant β, $0<\beta<1$.

Assumption 5. *$V_r(x)$ is locally integrable.*

Remark. Though Uchiyama did not refer to this assumption explicitly, it seems that we need some estimate for $|V_r(x)|$ in order to justify the integration by parts or the differentiation of forms. But, as he mentioned, if $V(x)$ is a homogeneous function of degree -2β satisfying Stummel's condition, then Assumptions 4 and 5 are satisfied. Particularly, the

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Coulomb potential of many-body problem

$$V(x)=\sum_{i<j}\frac{a_{ij}}{|x_i-x_j|}+\sum_i\frac{b_i}{|x_i|}$$

is such one with $\beta=1/2$.

Theorem. *If Assumptions 1∼5 are satisfied and if* $u(x)\in H^2_{\mathrm{loc}}(\Omega)$ *is a solution of* (1.1) *which does not vanish identically in a neighbourhood of infinity, then*

$$\liminf_{R\to\infty}\ R^{\beta-1}\int_{r_0\leq|x|\leq R}|u(x)|^2\,dx>0 \quad when \quad 0<\beta<1 \tag{1.2}$$

and

$$\liminf_{R\to\infty}\ (\log R)^{-1}\int_{r_0\leq|x|\leq R}|u(x)|^2\,dx>0 \quad when \quad \beta=1. \tag{1.3}$$

This theorem implies that, whatever boundary condition may be posed, $-\triangle+V(x)$ has no positive eigenvalues. To say more precisely, if the shape of Ω is moderate so as to assure the unique continuation property of the solution of (1.1), we can conclude the absence of positive eigenvalues of any selfadjoint realization of $-\triangle+V(x)$ in $L^2(\Omega)$, regardless of the boundary condition.

Uchiyama has proved the same estimate in the special case when $\Omega=R^n$. For $\Omega\neq R^n$, however, he showed (1.3) when $\beta=1$ and (1.2) when $\frac{1}{3}<\beta<1$. Our result is thus a direct extension of his. The proof is preformed by simpler calculations with the aid of S. Agmon's trick ([1]). There does not appear any critical value of β like 1/3 even in the process of the proof.

§ 2. Lemmas

We begin with reducing the problem to a differential equation in $L^2(S^{n-1})$. Set $v=r^{(n-1)/2}u$, for which we may have

$$v_{rr}+r^{-2}\Lambda v+(\lambda-q)v=0.$$

Here and in the sequel we write $v_r=\frac{\partial v}{\partial r}$, $v_{rr}=\frac{\partial^2 v}{\partial r^2}$, $\Lambda=$the Laplace-Beltrami operator on S^{n-1} and

$$q(x)=V(x)+\frac{(n-1)(n-3)}{4r^2}.$$

In what follows, we think of $v(x)=v(r,\ \cdot\ )$ as a function of r with values in $L^2(S^{n-1})$. Now, let us introduce a bilinear form

$$\begin{aligned}F(r)=&\|v_r\|^2+\lambda\|v\|^2-(qv,\ v)+r^{-2}(\Lambda v,\ v)\\&-\beta r^{-1}(v_r,\ v)+\frac{1}{2}\beta(\beta-1)r^{-2}\|v\|^2 \quad (r\geqq r_0),\end{aligned}$$

where $\|\ \ \|$ and $(\ ,\ )$ are the norm and the inner product in $L^2(S^{n-1})$. We note that $q(x)$ also fulfils Assumptions 2∼5. Since $u(x)$ belongs to $H^2_{\mathrm{loc}}(\Omega)$ and since we can show that $q_r u^2\in L^1_{\mathrm{loc}}(\Omega)$ (see the remark after Lemma 2), each term of the right-hand side is absolutely continuous with respect to r so that

$$\begin{aligned}F'(r)&=2(v_{rr},\ v_r)+2\lambda(v,\ v_r)-2(qv,\ v_r)-(q_rv,\ v)+2r^{-2}(\Lambda v,\ v_r)\\&\quad-2r^{-3}(\Lambda v,\ v)+\beta r^{-2}(v_r,\ v)-\beta r^{-1}\|v_r\|^2-\beta r^{-1}(v_{rr},\ v)\\&\quad+\beta(\beta-1)r^{-2}(v_r,\ v)-\beta(\beta-1)r^{-3}\|v\|^2\\&=2(v_{rr}+r^{-2}\Lambda v+(\lambda-q)v,\ v_r)-(q_rv,\ v)-2r^{-3}(\Lambda v,\ v)\\&\quad+\beta^2r^{-2}(v_r,\ v)-\beta r^{-1}\|v_r\|^2+\beta r^{-3}(\Lambda v,\ v)+\beta\lambda r^{-1}\|v\|^2\\&\quad-\beta r^{-1}(qv,\ v)-\beta(\beta-1)r^{-3}\|v\|^2\\&=-(q_rv,\ v)+(\beta-2)r^{-3}(\Lambda v,\ v)+\beta^2r^{-2}(v_r,\ v)-\beta r^{-1}\|v_r\|^2\\&\quad+\beta r^{-1}(\lambda-(\beta-1)r^{-2})\|v\|^2-\beta r^{-1}(qv,\ v),\end{aligned}$$

almost everywhere (we shall sometimes omit these words). Therefore,

$$\begin{aligned}(r^\beta F(r))'&=r^{\beta-1}(rF'(r)+\beta F(r))\\&=r^{\beta-1}\{-(rq_rv,\ v)+(\beta-2)r^{-2}(\Lambda v,\ v)+\beta^2r^{-1}(v_r,\ v)-\beta\|v_r\|^2\\&\quad+\beta(\lambda-(\beta-1)r^{-2})\|v\|^2-\beta(qv,\ v)+\beta\|v_r\|^2+\beta\lambda\|v\|^2-\beta(qv,\ v)\\&\quad+\beta r^{-2}(\Lambda v,\ v)-\beta^2r^{-1}(v_r,\ v)+\frac{1}{2}\beta^2(\beta-1)r^{-2}\|v\|^2\}\\&=r^{\beta-1}\{-([rq_r+2\beta q]v,\ v)+2(\beta-1)r^{-2}(\Lambda v,\ v)\\&\quad+\beta(2\lambda+\frac{1}{2}(1-\beta)(2-\beta)r^{-2})\|v\|^2\}.\end{aligned}$$

Hence we can establish the inequality

$$(r^\beta F(r))'\geqq 2\beta\lambda r^{\beta-2}\|v\|^2\geqq 0\qquad\text{a. e.}\tag{2.1}$$

Lemma 1. *There exists a value of r for which $F(r)>0$.*

Proof. Suppose that $F(r)\leqq 0$ for every $r\geqq r_0$. Then,

$$\begin{aligned}-r^\beta F(r)&=-t^\beta F(t)+\int_r^t(s^\beta F(s))'ds\\&\geqq 2\beta\lambda\int_r^t s^{\beta-1}\|v(s)\|^2ds\qquad(r_0\leqq r<t).\end{aligned}$$

Since the left side is independent of t, we can put $t\to\infty$ on the right side obtaining

$$-F(r)\geqq 2\beta\lambda r^{-\beta}\int_r^\infty s^{\beta-1}\|v(s)\|^2ds.\tag{2.2}$$

Now we put

$$g(r)=\log\|v(r)\|^2$$

on an interval I where $v(r)\neq 0$. Then, we have

$$\begin{aligned}g''(r)&=\frac{d}{dr}\ \frac{2(v_r,\ v)}{\|v\|^2}\\&=\frac{2}{\|v\|^2}\left\{(v_{rr},\ v)+\|v_r\|^2-\frac{2(v_r,\ v)^2}{\|v\|^2}\right\}\\&\geqq 2e^{-g(r)}\{(v_{rr},\ v)-\|v_r\|^2\}\\&=-2e^{-g(r)}\{\|v_r\|^2+r^{-2}(\Lambda v,v)+((\lambda-q)v,\ v)\}\\&=-2e^{-g(r)}\{F(r)+\beta r^{-1}(v_r,\ v)-\frac{1}{2}\beta(\beta-1)r^{-2}\|v\|^2\}\\&=-2e^{-g(r)}F(r)+\beta(\beta-1)r^{-2}-\beta r^{-1}g'(r),\end{aligned}$$

for almost every r since $g'(r)$ is absolutely continuous. Therefore we have

$$r^{-\beta}(r^{\beta}g'(r))'=g''(r)+\beta r^{-1}g'(r) \qquad (2.3)$$
$$\geqq -2e^{-g(r)}F(r)+\beta(\beta-1)r^{-2}.$$

Consequently, by the assumption that $F(r)\leqq 0$, we are led to

$$(r^{\beta}g'(r))'\geqq\beta(\beta-1)r^{\beta-2}$$

which implies

$$r^{\beta}g'(r)\geqq C+\beta r^{\beta-1}\geqq -C'$$

where C and C' are some constants with $C'>0$.

At first, we consider the case $0<\beta<1$. Then, in virtue of the above inequality, we have

$$g(s)-g(r)\geqq -C'\int_r^s t^{-\beta}ds=-\frac{C'}{1-\beta}(s^{1-\beta}-r^{1-\beta}) \quad (r,\ s\in I,\ \ r<s). \qquad (2.4)$$

This inequality implies that $g(s)$ can not tend to $-\infty$ at any finite point. But, since $\|v(s)\|^2$ is continuous, this statement would run in contradiction if there were a point where $\|v(s)\|=0$. Thus, $\|v(r)\|>0$ throughout $(r_0,\ \infty)$, or in other words, $I=(r_0,\ \infty)$. Now, (2.2) and (2.3) show

$$(r^{\beta}g'(r))'\geqq -2r^{\beta}e^{-g(r)}F(r)+\beta(\beta-1)r^{\beta-2}$$
$$\geqq 4\beta\lambda\int_r^{\infty}s^{\beta-1}e^{g(s)-g(r)}ds+\beta(\beta-1)r^{\beta-2}.$$

Writing $C=C'/(1-\beta)$ in (2.4), we have

$$(r^{\beta}g'(r))'\geqq 4\beta\lambda\int_r^{\infty}s^{\beta-1}e^{-C(s^{1-\beta}-r^{1-\beta})}ds+\beta(\beta-1)r^{\beta-2}.$$

Thus, setting $t=s^{1-\beta}$, we see

$$R^{\beta}g'(R)\geqq \text{const.}+4\beta\lambda\int_{r_0}^{R}e^{Cr^{1-\beta}}dr\int_r^{\infty}s^{\beta-1}e^{-Cs^{1-\beta}}ds+\beta R^{\beta-1}$$
$$=\text{const.}+\frac{4\beta\lambda}{1-\beta}\int_{r_0}^{R}e^{Cr^{1-\beta}}dr\int_{r^{1-\beta}}^{\infty}t^{\frac{1}{1-\beta}-2}e^{-Ct}dt+\beta R^{\beta-1}.$$

Let us notice that for any number γ,

$$\int_x^{\infty}t^{-\gamma}e^{-Ct}dt=\frac{1}{C}x^{-\gamma}e^{-Cx}-\frac{2}{C^2}x^{-\gamma-1}e^{-Cx}+\frac{\gamma(\gamma+1)}{C^2}\int_x^{\infty}t^{-\gamma-2}e^{-Ct}dt$$
$$\geqq\frac{1}{2C}x^{-\gamma}e^{-Cx} \qquad (\text{for large } x),$$

which leads to

$$R^{\beta}g'(R)\geqq \text{const.}+\frac{2\beta\lambda}{1-\beta}\cdot\frac{1}{C}\int_{r_0}^{R}r^{2\beta-1}dr+\beta R^{\beta-1}$$
$$\geqq\frac{\lambda}{2C(1-\beta)}R^{2\beta} \qquad (\text{for large } R).$$

But this implies

$$g'(r)\to\infty \quad (r\to\infty)$$

and hence

$$\|v(r)\|\to\infty \quad (r\to\infty)$$

which contradicts that $\int_r^\infty s^{\beta-1}\|v(s)\|^2 ds<\infty$. So we obtain the lemma for $0<\beta<1$.

If $\beta=1$, the inequality (2.3) reads

$$(rg'(r))'\geqq -2re^{-g(r)}F(r)\geqq 0.$$

Hence

$$rg'(r)\geqq -C$$

fore some constant $C>1$, and hence

$$g(s)-g(r)\geqq -C(\log s-\log r).$$

Thus, similarly to the previous case, $\|v(r)\|\neq 0$ on (r_0, ∞) and

$$\begin{aligned}(rg'(r))'&\geqq -2re^{-g(r)}F(r)\\ &\geqq 4\lambda\int_r^\infty e^{g(s)-g(r)}ds\\ &\geqq 4\lambda\int_r^\infty e^{-C(\log s-\log r)}ds\\ &=\frac{4\lambda}{C-1}r,\end{aligned}$$

hence

$$rg'(r)\geqq C_1+\frac{2\lambda}{C-1}r^2.$$

Therefore, we have $\|v(r)\|\to\infty$ $(r\to\infty)$ again, which contradicts $\int_r^\infty \|v(s)\|^2 ds<\infty$. (Lemma 1 is proved.)

This lemma gives the estimate of $\int_{r_0}^R F(r)dr$ from below. To obtain the estimate of $\int_{r_0}^R \|v(r)\|^2 dr$, however, we need one more lemma (Lemma 3) which is essentially the same as Lemma 3.16 of Uchiyama [5]. We shall repeat its proof in order to bring it into line with our settings and notations. For this purpose, we refer to the following lemma due to Ikebe-Kato [3; Lemma 2 summarized for our case]:

Lemma 2. *If $q(x)$ satisfies Stummel's condition, then for any $u\in H^1{}_{\rm loc}(R^n)$, the product $|q|^{\frac12}u$ belongs to $L^2{}_{\rm loc}(R^n)$, and for any R and δ $(R>0,\ 0<\delta<1)$ we have*

$$\int_{|x|\leqq R}|q(x)|u(x)^2dx\leqq \text{const.}\,\delta^{\frac12}\int_{|x|\leqq R+\delta}|\nabla u(x)|^2dx+\text{const.}\,\delta^{-2+\frac12\alpha}\int_{|x|\leqq R+\delta}|u(x)|^2dx.$$

Here the constants depend only on q and n.

Remark. By modifying a similar inequality described in [3], we can obtain

$$\begin{aligned}\int_{r_0\leqq|x|\leqq R}|q_r(x)|u(x)^2dx\leqq &\text{ const.}\,\delta^{-n+4}\int_{r_0-\delta\leqq|x|\leqq R+\delta}|\triangle u(y)|^2dy\int_{|x-y|\leqq\delta}|q_r(x)|dx\\ &+\text{const.}\,\delta^n\int_{r_0-\delta\leqq|x|\leqq R+\delta}|u(y)|^2dy\int_{|x-y|\leqq\delta}|q_r(x)|dx\end{aligned}$$

for any $u(x)\in H^2{}_{\rm loc}(\Omega)$. Therefore, if $q_r(x)\in L^1{}_{\rm loc}(\Omega)$, then $q_r(x)u(x)^2\in L^1{}_{\rm loc}(\Omega)$. This fact implies the absolute continuity of

$$(qv, v)=\int_{r_0}^r dr\int_{S^{n-1}}q_r(x)v(x)^2d\omega+2\int_{r_0}^r dr\int_{S^{n-1}}q(x)v(x)v_r(x)d\omega+\text{const.}$$

and gives justification of the formula

$$\frac{d}{dr}(qv,\ v)=(q_r v,\ v)+2(qv,\ v_r) \qquad \text{a. e.}$$

Let us now consider a C^2-function $\zeta_R(r)$ with the properties (i) $0 \leqq \zeta_R(r) \leqq 1$ $(r_0 \leqq r < \infty)$ (ii) $\zeta_R(r)=0$ $(r_0 \leqq r \leqq r_0+0.1)$ (iii) $\zeta_R(r)=1$ $(r_0+1 \leqq r \leqq R-1)$ (iv) $\zeta_R(r)=0$ $(R-0.1 \leqq r < \infty)$ (v) In $(r_0,\ r_0+1)$ the value of $\zeta_R(r)$ does not depend on R and in $(R-1,\ R)$ the graph does not change its shape but for translation. (Imagine a tidal wave or a slide-trombone.)

Lemma 3. For an arbitrarily small $\varepsilon>0$, there exist constants C_ε and M_ε such that

$$\int_{r_0}^R \zeta_R(r)^2 |(qv,\ v)|\, dr \leqq \int_{r_0}^R \zeta_R(r)^2 (2\varepsilon \|v_r\|^2 + \varepsilon r^{-2} \|\nabla' v\|^2)\, dr$$
$$+\int_{r_0}^R (2\varepsilon \zeta_R'(r)^2 + C_\varepsilon \zeta_R(r)^2 + \varepsilon r^{-2} \zeta_R(r)^2) \|v\|^2 dr \qquad (2.5)$$

and

$$\int_{r_0}^R \zeta_R(r)^2 \|v_r\|^2 dr \leqq M_\varepsilon \int_{r_0}^R \|v\|^2 dr - \frac{1-\varepsilon}{1-2\varepsilon} \int_{r_0}^R \zeta_R(r)^2 r^{-2} \|\nabla' v\|^2 dr, \qquad (2.6)$$

where $\nabla' v$ denotes the gradient of v on S^{n-1}, in other words, $\|\nabla' v\|^2 = -(\Lambda v,\ v)$.

Proof. We apply Lemma 2 to $\zeta_R(r) u = \zeta_R(r) r^{-(n-1)/2} v$. At first we note that $|\nabla f|^2 = f_r^2 + r^{-2} |\nabla' f|^2$ and hence that

$$|\nabla(\zeta_R u)|^2 = r^{-(n-1)} (\zeta_R v)_r^2 - (n-1) r^{-n} (\zeta_R v)_r \zeta_R v$$
$$+\frac{(n-1)^2}{4} \zeta_R^2 r^{-(n+1)} v^2 + \zeta_R^2 r^{-(n+1)} |\nabla' v|^2.$$

Then, Lemma 2 shows

$$\int_{r_0}^R \zeta_R^2 |(qv,\ v)|\, dr \leqq \text{const.}\ \delta^{\frac{1}{2}\alpha} \int_{r_0}^R \{\|(\zeta_R v)_r\|^2 - (n-1) r^{-1} ((\zeta_R v)_r,\ \zeta_R v)$$
$$+\frac{(n-1)^2}{4} \zeta_R^2 r^{-2} \|v\|^2 + \zeta_R^2 r^{-2} \|\nabla' v\|^2\}\, dr$$
$$+\text{const.}\ \delta^{-2+\frac{1}{2}\alpha} \int_{r_0}^R \zeta_R^2 \|v\|^2 dr$$
$$\leqq \text{const.}\ \delta^{\frac{1}{2}\alpha} \int_{r_0}^R \left\{\frac{3}{2} \|(\zeta_R v)_r\|^2 + \frac{3}{4} (n-1)^2 \zeta_R^2 r^{-2} \|v\|^2\right.$$
$$\left. + \zeta_R^2 r^{-2} \|\nabla' v\|^2\right\} dr + \text{const.}\ \delta^{-2+\frac{1}{2}\alpha} \int_{r_0}^R \zeta_R^2 \|v\|^2 dr,$$

where we can choose δ arbitrarily small. Setting the constants appropriately, we have

$$\int_{r_0}^R \zeta_R^2 |(qv,\ v)|\, dr \leqq \varepsilon \int_{r_0}^R (\|(\zeta_R v)_r\|^2 + \zeta_R^2 r^{-2} \|v\|^2$$
$$+\zeta_R^2 r^{-2} \|\nabla' v\|^2)\, dr + C_\varepsilon \int_{r_0}^R \zeta_R^2 \|v\|^2 dr.$$

Note that

$$\|(\zeta_R v)_r\|^2 = \zeta_R^2 \|v_r\|^2 + 2\zeta_R \zeta_R' (v_r,\ v) + \zeta_R'^2 \|v\|^2 \leqq 2\zeta_R^2 \|v_r\|^2 + 2\zeta_R'^2 \|v\|^2,$$

then we can obtain the inequality (2.5) immediately.

Now we turn to the equation $v_{rr} + r^{-2} \Lambda v + (\lambda - q) v = 0$, which gives

$$\int_{r_0}^R \zeta_R^2 \{2\|v_r\|^2 - 2((\lambda - q) v,\ v)\}\, dr$$
$$= \int_{r_0}^R \zeta_R^2 \{2\|v_r\|^2 + 2(v_{rr},\ v) + 2r^{-2} (\Lambda v,\ v)\}\, dr$$

$$=\int_{r_0}^{R}\zeta_R^2\frac{d^2}{dr^2}\|v\|^2dr-2\int_{r_0}^{R}\zeta_R^2r^{-2}\|\nabla'v\|^2dr$$

$$=\int_{r_0}^{R}(\zeta_R^2)''\|v\|^2dr-2\int_{r_0}^{R}\zeta_R^2r^{-2}\|\nabla'v\|^2dr.$$

Hence, combining this with (2.5), we have

$$\int_{r_0}^{R}\zeta_R^2\|v_r\|^2dr=\frac{1}{2}\int_{r_0}^{R}(\zeta_R^2)''\|v\|^2dr-\int_{r_0}^{R}\zeta_R^2r^{-2}\|\nabla'v\|^2dr$$

$$+\int_{r_0}^{R}\zeta_R^2((\lambda-q)v,\ v)dr$$

$$\leqq\frac{1}{2}\int_{r_0}^{R}(\zeta_R^2)''\|v\|^2dr-\int_{r_0}^{R}\zeta_R^2r^{-2}\|\nabla'v\|^2dr$$

$$+\int_{r_0}^{R}\zeta_R^2\lambda\|v\|^2dr+2\varepsilon\int_{r_0}^{R}\zeta_R^2\|v_r\|^2dr+\varepsilon\int_{r_0}^{R}\zeta_R^2r^{-2}\|\nabla'v\|^2dr$$

$$+\int_{r_0}^{R}(2\varepsilon\zeta_R'^2+C_\varepsilon\zeta_R^2+\varepsilon r^{-2}\zeta_R^2)\|v\|^2dr.$$

Thus we obtain (2.6) with

$$M_\varepsilon=\frac{1}{1-2\varepsilon}\sup_{r_0\leqq r}\left|\frac{1}{2}(\zeta_R^2)''+2\varepsilon\zeta_R'^2+(\lambda+C_\varepsilon+\varepsilon r^{-2})\zeta_R^2\right|$$

which is clearly seen to be independent of R. (Lemma 3 is proved.)

§ 3. Proof of the theorem.

We note that no generality will be lost if we think of the value of r in Lemma 1 as r_0 itself, because the desired inequality in the theorem does not affected by changing the constant r_0. Therefore, Lemma 1 together with (2.1) shows

$$r^\beta F(r)\geqq r_0^\beta F(r_0)>0 \qquad (r_0\leqq r)$$

and hence

$$\int_{r_0+1}^{R-2}F(r)dr\geqq CR^{1-\beta} \qquad (\text{if } 0<\beta<1)$$

or

$$\int_{r_0+1}^{R-1}F(r)dr\geqq C\log R \qquad (\text{if } \beta=1)$$

for some positive constant C.

On the other hand,

$$\int_{r_0+1}^{R-1}F(r)dr=\int_{r_0+1}^{R-1}\Big\{\|v_r\|^2+\lambda\|v\|^2-(qv,\ v)-r^{-2}\|\nabla'v\|^2$$

$$-\beta r^{-1}(v_r,\ v)+\frac{1}{2}\beta(\beta-1)r^{-2}\|v\|^2\Big\}dr$$

$$\leqq\int_{r_0+1}^{R-1}\Big\{\frac{3}{2}\|v_r\|^2+\lambda\|v\|^2+|(qv,\ v)|+\frac{1}{2}\beta r^{-2}\|v\|^2\Big\}dr$$

$$\leqq\int_{r_0}^{R}\Big\{\frac{3}{2}\zeta_R^2\|v_r\|^2+\zeta_R^2(\lambda+\beta r^{-2})\|v\|^2+\zeta_R^2|(qv,\ v)|\Big\}dr.$$

Hence, by dint of (2.5) and (2.6) we have

$$\int_{r_0+1}^{R-1}F(r)dr\leqq\int_{r_0}^{R}\Big(\frac{3}{2}+2\varepsilon\Big)\zeta_R^2\|v_r\|^2dr$$

$$+\int_{r_0}^{R}\left\{\left(\lambda+\frac{1}{2}\beta r^{-2}+\varepsilon r^{-2}+C_\varepsilon\right)\zeta_R{}^2+2\varepsilon\zeta_R{}'^2\right\}\|v\|^2dr$$

$$+\varepsilon\int_{r_0}^{R}\zeta_R{}^2r^{-2}\|\nabla' v\|^2dr$$

$$\leqq M_\varepsilon\left(\frac{3}{2}+2\varepsilon\right)\int_{r_0}^{R}\|v\|^2dr+\text{const.}\int_{r_0}^{R}\|v\|^2dr$$

$$-\left(\frac{1-\varepsilon}{1-2\varepsilon}-\varepsilon\right)\int_{r_0}^{R}\zeta_R{}^2r^{-2}\|\nabla' v\|^2dr.$$

If $0<\varepsilon<1/2$ then $\frac{1-\varepsilon}{1-2\varepsilon}-\varepsilon>0$ and we obtain

$$\int_{r_0+1}^{R-1}F(r)dr\leqq\text{const.}\int_{r_0}^{R}\|v\|^2dr$$

and hence

$$\int_{r_0}^{R}\|v\|^2dr\geqq\text{const. }R^{1-\beta}\qquad(\text{if }0<\beta<1)$$

or

$$\int_{r_0}^{R}\|v\|^2dr\geqq\text{const. }\log R\qquad(\text{if }\beta=1)$$

which yields the conclusion of the theorem.

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