Dimension Formula for the Spaces of Siegel Cusp Forms and a Certain Exponential Sum[1]

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Synopsis. In this paper we present an explicit dimension formula for the spaces of Siegel cusp forms of degree two with respect to the congruence subgroup $\Gamma_0(p)$ and Dirichlet character modulo p. A certain exponential sum appears in the formula. The value of this sum is determined by comparing our result with the result of K. Hashimoto ([Ha1])[3].

Keyword. Siegel cusp form, holomorphic Lefschetz fixed point formula, exponential sum

Contents

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[3] After this paper was accepted, T. Ibukiyama directly evaluated the exponential sum.

Notations

$\mathfrak{S}_g$: the Siegel upper half plane of degree g

$Sp\,(g,\mathbf{R}), Sp\,(g,\mathbf{Z})$: the symplectic group over $\mathbf{R}$ and $\mathbf{Z}$

$M\langle Z\rangle$: $(AZ+B)(CZ+D)^{-1}$

$J(M,Z)$: $CZ+D$ (the canonical automorphic factor)

μ : (irreducible) holomorphic representation of $GL(2,\mathbf{C})$ into $GL(r,\mathbf{C})$

Γ : a subgroup of finite index of $Sp\,(g,\mathbf{Z})$

$M_\mu(\Gamma)$ $(S_\mu(\Gamma))$: the space of automorphic (cusp) forms of type μ with respect to Γ

$M_k(\Gamma)$ $(S_k(\Gamma))$: the space of automorphic (cusp) forms of weight k with respect to Γ

$\Gamma_0(N)$: the subgroup of $Sp\,(2,\mathbf{Z})$ defined by $C \equiv O \pmod N$ (§0)

$\Gamma_g(N)$: the principal congruence subgroup of level N of $Sp\,(g,\mathbf{Z})$

$S_\mu(\Gamma_0(N),\chi)$: the space of cusp forms of type μ w.r.t. $\Gamma_0(N)$ and Dirichlet character χ (§0)

$\left(\dfrac{a}{p}\right)$: Legendre symbol

$h(-p)$: the class number of $\mathbf{Q}(\sqrt{-p})$

X^g, X^g_α : the set of fixed points of g and its irreducible component

$\tau(g,X^g_\alpha), \tau(g)$: the contribution of g at the fixed point set X^g_α and their sum (§1)

Φ : a representative of the irreducible components of the fixed points sets

$C_G(\Phi), N_G(\Phi)$: the isotropy group and the stabilizer group in G of Φ (Definition 1.4)

$C^p_G(\Phi)$: the set of proper elements of $C_G(\Phi)$ (Definition 1.4)

$C_G(g)$: the centralizer group in G of an element g of G

$e(\varphi)$: (§1)

$X_2(N)$: the quotient space $\Gamma_2(N)\backslash\mathfrak{S}_2$

$\overline{X}_2(N)$: the Satake compactification of $X_2(N)$

$\widetilde{X}_2(N)$: the smooth compactification of $X_2(N)$ $(N \geq 3)$

$D = \widetilde{X}_2(N) - X_2(N)$: the divisor at infinity

$[D]$: the line bundle associated with the divisor D

$s : \widetilde{X}_2(N) \to \overline{X}_2(N)$: the map of $\widetilde{X}_2(N)$ which is the identity on $X_2(N)$

$G(N), G_0(N)$: $\Gamma_2(1)/\Gamma_2(N), \Gamma_0(N)/\Gamma_2(N)$

$\mathcal{V}_\mu$: $\mathfrak{S}_2 \times \mathbf{C}^r$ with the action $M(Z,v) = (M\langle Z\rangle, \mu(CZ+D)v)$ of $M \in \Gamma_2(1)$

$\mathcal{L}_g$: $\mathfrak{S}_g \times \mathbf{C}$ with the action $M(Z,v) = (M\langle Z\rangle, \det(CZ+D)v)$ of $M \in \Gamma_g(1)$

V_μ : the vector bundle $\Gamma_2(N)\backslash\mathcal{V}_\mu$ over $X_2(N)$ $(N \geq 3)$

$\widetilde{V}_\mu$: the extension of V_μ onto $\widetilde{X}_2(N)$ $(N \geq 3)$

L_g : the line bundle $\Gamma_g(N)\backslash\mathcal{L}_g$ over $X_g(N)$ $(N \geq 3)$

$\widetilde{L}_g, \overline{L}_g$: the extension of L_g onto $\widetilde{X}_g(N)$ and $\overline{X}_g(N)$ $(N \geq 3)$

s_j : the symmetric tensor representation of degree j of $GL(2,\mathbf{C})$

$\mathcal{V}, V, \widetilde{V}$: $\mathcal{V}_\mu, V_\mu, \widetilde{V}_\mu,$ where μ is the standard action of $GL(2,\mathbf{C})$ on $\mathbf{C}^2$

D_1 : irreducible component of D

π : the restriction of s to D_1

C : $\pi(D_1)$ (one dimensional cusp of $\overline{X}_2(N)$)

$\mathcal{H}$: the hermitian metric on $\mathcal{V}$ defined by $\mathrm{Im}Z$

h : the hermitian metric on V induced by $\mathcal{H}$

θ, Θ : the connection form and the curvature form of h

$\mathbf{P}(V)$: the projective bundle associated with vector bundle V

$H(V)$: the dual line bundle of the tautological line bundle of $\mathbf{P}(V)$

ϖ : the projection: $\mathbf{P}(\widetilde{V}) \to \widetilde{X}_2(N)$ $(N \geq 3)$
$\widehat{h}$: the hermitian metric on $H(V)^*$ induced by h

§0. Introduction

Let $\mathfrak{S}_g = \{Z \in M_g(\mathbf{C}) \mid {}^tZ = Z,\ \mathrm{Im}\ Z > 0\}$ be the Siegel upper half plane of degree g. Let

$$Z \in \mathfrak{S}_g \text{ and } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp\,(g, \mathbf{R}),$$

and let

$$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}.$$

Then this defines an action of $Sp\,(g, \mathbf{R})$ on $\mathfrak{S}_g$. Let Z and M be as above, and put

$$J(M, Z) = CZ + D\ (\in GL(g, \mathbf{C})).$$

Then this satisfies the following relation for any $M_1, M_2 \in Sp\,(g, \mathbf{R})$ and $Z \in \mathfrak{S}_g$:

$$J(M_1M_2, Z) = J(M_1, M_2\langle Z\rangle)J(M_2, Z),$$

and this is called *the canonical automorphic factor*. Let μ be a holomorphic representation of $GL(g, \mathbf{C})$ into $GL(r, \mathbf{C})$. Then $\mu(J(M, Z)) = \mu(CZ + D)$ also satisfies the above relation.

Let μ be as above and let Γ be a discrete subgroup of $Sp\,(g, \mathbf{R})$. By *an automorphic form of type* μ with respect to Γ, we mean a holomorphic map f of $\mathfrak{S}_g$ to the r-dimensional complex vector space $\mathbf{C}^r$ which satisfies the following equalities:

$$f(M\langle Z\rangle) = \mu(CZ + D)f(Z),$$

for any $M \in \Gamma$ and $Z \in \mathfrak{S}_g$. (We need to assume the holomorphy of f at the "cusps" if $g = 1$.) We denote by $M_\mu(\Gamma)$ the complex vector space of automorphic forms of type μ with respect to Γ. If Γ is arithmetically defined discrete subgroup of $Sp\,(g, \mathbf{R})$, then it is known that $M_\mu(\Gamma)$ is finite-dimensional.

Let Γ be a subgroup of finite index of $Sp\,(g, \mathbf{Z})$. An automorphic form f of type μ with respect to Γ is called *a cusp form* if it belongs to the kernel of Φ-operator ([Gd] and Definition 5.7, below). We denote by $S_\mu(\Gamma)$ the vector space of cusp forms of type μ with respect to Γ. In case $\mu(CZ + D) = \det(CZ + D)^k$, an automorphic (resp. a cusp) form of type μ is also called an automorphic (resp. a cusp) form of *weight* k, and $M_\mu(\Gamma)$ (resp. $S_\mu(\Gamma)$) is also denoted by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$).

Now let $g=2$, and for a natural number N let

$$\Gamma_0(N)=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp\,(2,\mathbf{Z})\,\middle|\, C\equiv O \pmod N\right\}.$$

Let χ be a Dirichlet character modulo N. We denote by $S_\mu(\Gamma_0(N),\chi)$ the vector space of the holomorphic maps f of $\mathfrak{S}_2$ to $\mathbf{C}^r$ which satisfy the following equalities:

$$f(M\langle Z\rangle)=\chi(\det(D))\mu(CZ+D)f(Z),$$

for any $M\in\Gamma_0(N)$ and $Z\in\mathfrak{S}_2$ and belong to the kernel of Φ-operator. In case χ is the trivial character, $S_\mu(\Gamma_0(N),\chi)$ is equal to $S_\mu(\Gamma_0(N))$. For a prime number p the dimension of $S_\mu(\Gamma_0(p),\chi)$ is calculated in this paper by using the holomorphic Lefschetz fixed point formula.

Among the contributions of the fixed points at infinity, the following exponential sum which includes the Legendre symbol appears in our calculations:

$$\sum\left(\frac{s^2-rt}{p}\right)(\zeta^{r+s}-1)^{-1}(\zeta^{s+t}-1)^{-1}(\zeta^{-s}-1)^{-1},$$

where $\zeta=\exp(2\pi\sqrt{-1}/p)$ and $(r,s,t)\in\mathbf{F}_p^3$ is over the triple such that $s(r+s)(s+t)\neq 0$ and $s^2\neq rt$. The author computed this exponential sum by using computer for primes such that $5\le p<500$ and found this sum is equal to

$$-\frac{p(p-1)^2}{8}+\begin{cases} 0, & if\ p\equiv 1 \pmod 4 \\ \frac{3}{2}ph(-p)^2, & if\ p\equiv 3 \pmod 4 \end{cases},$$

where $h(-p)$ is the class number of $\mathbf{Q}(\sqrt{-p})$. But we did not prove the above equality for general primes. In the case of weight k and trivial character, K. Hashimoto calculated the dimension of $S_k(\Gamma_0(p))$ by using the Selberg's trace formula ([Ha1]). By comparing our formula with Hashimoto's result, we know the above equality holds for general primes. A similar exponential sum appeared in [T4] concerning a representation of $Sp\,(2,\mathbf{F}_p)$ and the author presented a conjecture. This conjecture was also proved by K. Hashimoto by using the Selberg's trace formula in [Ha2].

Recently the author calculated the dimension of the spaces of Siegel cusp forms of half integral weight and degree two ([T5]). When one study the possibility of "Shimura correspondence" (cf. [Shm], [Shn] for elliptic modular case) in the case of Siegel cusp forms, it will be very important to calculate the dimension of the related spaces of Siegel cusp forms and compare them. If we calculate them by using the holomorphic Lefschetz fixed point formula, similar exponential sums will appear and it will be necessary to evaluate them. So it is desirable to prove the above equality directly.

Let $\Gamma_g(N) = \{M \in Sp\,(g, \mathbf{Z}) \mid M \equiv 1_{2g} \pmod N\}$ be the principal congruence group of level N ($N \geq 1$) of $Sp\,(g, \mathbf{Z})$. The author calculated the dimension of $S_\mu(\Gamma_2(N))$ ([T3]) but did not publish the details of it. In §3, §5 and §6, we present them. This result of $\dim S_\mu(\Gamma_2(N))$ has applications in [A1] and in [Sto].

§1. General Dimension Formula

Let X be a compact complex manifold and V a holomorphic vector bundle of rank n on X, and let G be a finite group of automorphism of the pair (X, V). Let G_0 be a subgroup of G and χ a character of G_0. We denote by $S(G_0, \chi)$ the vector space of the global holomorphic sections σ of V which satisfy

$$\sigma(g(x)) = \chi(g)g(\sigma(x)),$$

for any $g \in G_0$ and $x \in X$. In this section we present a general formula which represents the dimension of $S(G_0, \chi)$ by using the holomorphic Lefschetz fixed point formula.

First we recall the holomorphic Lefschetz fixed point formula. For $g \in G$ let X^g be the set of fixed points of g. X^g is a disjoint union of submanifolds of X. Let

$$X^g = \sum_\alpha X^g_\alpha$$

be the irreducible decomposition of X^g, and let

$$N^g_\alpha = \sum_\theta N^g_\alpha(\theta)$$

denote the normal bundle of X^g_α decomposed according to the eigenvalues $e^{i\theta}$ of g. We put

$$\begin{aligned} \mathcal{U}^\theta(N^g_\alpha(\theta)) &= \prod_\beta \left(\frac{1 - e^{-x_\beta - i\theta}}{1 - e^{-i\theta}}\right)^{-1} \\ &= \prod_\beta \left(1 + \frac{1}{1 - e^{i\theta}}x_\beta + \frac{1 + e^{i\theta}}{2(1 - e^{i\theta})^2}x_\beta^2 + \cdots\right), \end{aligned}$$

where the Chern class of $N^g_\alpha(\theta)$ is

$$c(N^g_\alpha(\theta)) = \prod_\beta (1 + x_\beta).$$

Let $\mathcal{T}(X^g_\alpha)$ be the Todd class of X^g_α. Let $V|X^g_\alpha$ be the restriction of V to X^g_α and $ch(V|X^g_\alpha)(g)$ the Chern character of $V|X^g_\alpha$ with g-action ([AS]). Put

$$\tau(g, X^g_\alpha) = \left\{ \frac{ch(V|X^g_\alpha)(g) \cdot \prod_\theta \mathcal{U}^\theta(N^g_\alpha(\theta)) \cdot \mathcal{T}(X^g_\alpha)}{\det(1 - g|(N^g_\alpha)^*)} \right\} [X^g_\alpha],$$

and

$$\tau(g) = \sum_\alpha \tau(g, X^g_\alpha).$$

Then we have

Theorem 1.1. (Holomorphic Lefschetz Fixed Point Formula [AS]).

$$\sum_{i \geq 0} (-1)^i \operatorname{trace}(g \mid H^i(X, \mathcal{O}(V))) = \tau(g).$$

Let V_x be the fiber of V at $x \in X$ and $g : V_x \to V_{g(x)}$ the action of g on V. Assume that g belongs to G_0. Then we denote by g_χ the action of g on V defined by $g_\chi(v) = \chi(g)g(v)$ $(v \in V_x)$. We call this action of G_0 on V the "twisted" action of G_0 by χ. We denote by V_χ the vector bundle V equipped with this action of G_0. Replacing $ch(V|X^g_\alpha)(g)$ by $\chi(g)ch(V|X^g_\alpha)(g)$ in the definition of $\tau(g, X^g_\alpha)$, we define $\tau(g_\chi, X^g_\alpha)$ and also we define $\tau(g_\chi)$ to be the sum of $\tau(g_\chi, X^g_\alpha)$'s. Since $ch(V_\chi|X^g_\alpha)(g_\chi) = \chi(g)ch(V|X^g_\alpha)(g)$, it holds that $\tau(g_\chi, X^g_\alpha) = \chi(g)\tau(g, X^g_\alpha)$. Hence we have $\tau(g_\chi) = \chi(g)\tau(g)$ and

Theorem 1.2. *If g belongs to G_0, then*

$$\sum_{i \geq 0} (-1)^i \operatorname{trace}(g_\chi \mid H^i(X, \mathcal{O}(V_\chi))) = \chi(g)\tau(g).$$

Let $H^i(X, \mathcal{O}(V_\chi))^{G_0}$ be the invariant subspace of $H^i(X, \mathcal{O}(V_\chi))$ by G_0. Then we have the following

Theorem 1.3.

$$\sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{O}(V_\chi))^{G_0} = \frac{1}{|G_0|} \sum_{g \in G_0} \chi(g)\tau(g).$$

Let $g, g' \in G$ and let X^g_α and $X^{g'}_{\alpha'}$ be irreducible components of X^g and $X^{g'}$, respectively. We define X^g_α and $X^{g'}_{\alpha'}$ to be *equivalent* if and only if there exists an element h of G which maps X^g_α to $X^{g'}_{\alpha'}$ biholomorphically. By this equivalence, we classify the irreducible components of the fixed points sets of G. Let Φ_λ $(\lambda \in \Lambda)$ be the representatives with respect to this equivalence.

Definition 1.4. Let $C_G(\Phi_\lambda)$ and $N_G(\Phi_\lambda)$ be the isotropy group and the stabilizer group of Φ_λ, respectively. Namely, we have

$$
\begin{aligned}
C_G(\Phi_\lambda) &= \{g \in G \mid g(x) = x \text{ for any } x \in \Phi_\lambda\}, \\
N_G(\Phi_\lambda) &= \{g \in G \mid g(\Phi_\lambda) = \Phi_\lambda\}.
\end{aligned}
$$

Let g be an element of $C_G(\Phi_\lambda)$. If Φ_λ is an irreducible component of X^g, g is called *a proper element* of $C_G(\Phi_\lambda)$. We denote by $C_G^p(\Phi_\lambda)$ the set of proper elements of $C_G(\Phi_\lambda)$.

Let $g \in G$. We have to count the number of irreducible components of X^g which are equivalent to Φ_λ $(\lambda \in \Lambda)$. $h(\Phi_\lambda)$ $(h \in G)$ is an irreducible component of X^g, if and only if $\varphi = h^{-1}gh$ belongs to $C_G^p(\Phi_\lambda)$. Let $C_G(\varphi)$ be the centralizer of φ in G. If h' belongs to $C_G(\varphi)$, it also holds that $\varphi = (hh')^{-1}g(hh')$. Hence $(hh')(\Phi_\lambda)$ is also an irreducible component of X^g. The number of irreducible components of X^g on which g acts as φ acts on Φ_λ is

$$n(\varphi) := \frac{|C_G(\varphi)|}{|C_G(\varphi) \cap N_G(\Phi_\lambda)|}.$$

The map

$$
\begin{aligned}
N_G(\Phi_\lambda) &\longrightarrow C_G(\Phi_\lambda) \\
g &\longmapsto g^{-1}\varphi g
\end{aligned}
$$

induces an injection of $N_G(\Phi_\lambda)/(C_G(\varphi) \cap N_G(\Phi_\lambda))$ to $C_G(\Phi_\lambda)$. The image of this map consists of elements of $C_G^p(\Phi_\lambda)$ which are conjugate to φ in $N_G(\Phi_\lambda)$. We denote by $e(\varphi)$ the number of the elements of this image and by $\equiv$ this conjugacy relation among the elements of $C_G^p(\Phi_\lambda)$ in $N_G(\Phi_\lambda)$. We have

$$
\begin{aligned}
n(\varphi) &= \frac{|C_G(\varphi)|}{|N_G(\Phi_\lambda)|} \cdot \frac{|N_G(\Phi_\lambda)|}{|C_G(\varphi) \cap N_G(\Phi_\lambda)|} \\
&= \frac{|C_G(\varphi)|}{|N_G(\Phi_\lambda)|} \cdot e(\varphi).
\end{aligned}
$$

Let $C_G^p(\Phi_\lambda)/\equiv$ be the set of the representatives of $C_G^p(\Phi_\lambda)$ classified by the relation $\equiv$. We denote by $\sim$ the conjugacy relation in G. If $\varphi, \varphi' \in C_G^p(\Phi_\lambda)$ satisfy $\varphi \equiv \varphi'$, it holds that $\tau(\varphi, \Phi_\lambda) = \tau(\varphi', \Phi_\lambda)$. Therefore we proved

Theorem 1.5.

$$\begin{aligned}\tau(g) &= \sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda)/\equiv,\varphi\sim g}\frac{|C_G(\varphi)|}{|N_G(\Phi_\lambda)|}\cdot e(\varphi)\cdot\tau(\varphi,\Phi_\lambda)\\ &= \sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda),\varphi\sim g}\frac{|C_G(\varphi)|}{|N_G(\Phi_\lambda)|}\cdot\tau(\varphi,\Phi_\lambda).\end{aligned}$$

We denote by $\approx$ the conjugacy relation in G_0 and by $G_0/\approx$ the set of the representatives of G_0 classified by the relation $\approx$. For $g\in G_0$, we denote by $C_{G_0}(g)$ the centralizer of g in G_0. If $g,g'\in G_0$ satisfy $g\approx g'$, it holds that $\mathrm{trace}\,(g\mid H^i(X,\mathcal{O}(V_\chi)))=\mathrm{trace}\,(g'\mid H^i(X,\mathcal{O}(V_\chi)))$. Hence by Theorem 1.3, we have

$$\begin{aligned}\sum_{i\geq 0}(-1)^i\dim H^i(X,\mathcal{O}(V_\chi))^{G_0} &= \frac{1}{|G_0|}\sum_{g\in G_0}\chi(g)\tau(g)\\ &= \frac{1}{|G_0|}\sum_{g\in G_0/\approx}\frac{|G_0|}{|C_{G_0}(g)|}\cdot\chi(g)\tau(g)\\ &= \sum_{g\in G_0/\approx}\frac{1}{|C_{G_0}(g)|}\cdot\chi(g)\tau(g).\end{aligned}$$

By Theorem 1.5 and above equality, we have

Theorem 1.6.

$$\begin{aligned}&\sum_{i\geq 0}(-1)^i\dim H^i(X,\mathcal{O}(V_\chi))^{G_0}\\ &= \sum_{g\in G_0/\approx}\frac{\chi(g)}{|C_{G_0}(g)|}\sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda),\varphi\sim g}\frac{|C_G(\varphi)|}{|N_G(\Phi_\lambda)|}\cdot\tau(\varphi,\Phi_\lambda)\\ &= \sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda)}\frac{\tau(\varphi,\Phi_\lambda)}{|N_G(\Phi_\lambda)|}\cdot\left(\sum_{g\in G_0/\approx,g\sim\varphi}\frac{|C_G(g)|}{|C_{G_0}(g)|}\cdot\chi(g)\right).\end{aligned}$$

$S(G_0,\chi)$ is canonically identified with $H^0(X,\mathcal{O}(V_\chi))^{G_0}$. Hence we have

Corollary 1.7. *If* $H^i(X,\mathcal{O}(V))\simeq 0$ *for all* $i>0$*, then*

$$\dim S(G_0,\chi)=\sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda)}\frac{\tau(\varphi,\Phi_\lambda)}{|N_G(\Phi_\lambda)|}\cdot\left(\sum_{g\in G_0/\approx,g\sim\varphi}\frac{|C_G(g)|}{|C_{G_0}(g)|}\cdot\chi(g)\right).$$

Remark 1.8. Instead of the last expression, it is sometimes convenient to use the following expression:

$$\sum_{\lambda\in\Lambda}\sum_{\varphi\in C_G^p(\Phi_\lambda)/\equiv}\frac{\tau(\varphi,\Phi_\lambda)}{|N_G(\Phi_\lambda)|}\cdot e(\varphi)\cdot\left(\sum_{g\in G_0/\approx,g\sim\varphi}\frac{|C_G(g)|}{|C_{G_0}(g)|}\cdot\chi(g)\right).$$

Remark 1.9. If one classified the fixed points sets and their isotropy groups and obtained the values of $|N_G(\Phi_\lambda)|$'s and $\tau(\varphi,\Phi_\lambda)$'s, he can calculate the dimension of $S(G_0,\chi)$ only classifying the conjugacy classes of G and G_0. Namely he may forget all of the geometric information. In the following of this paper, we proceed according to this principle. We give no geometric information about Φ_λ's nor φ's in this paper and postpone geometric arguments until §5.

§2. Conjugacy Classes of $\Gamma_0(p)/\Gamma_2(p)$

Let Γ be a subgroup of finite index of $Sp\,(g,\mathbf{Z})$. If $g\geq 2$, Γ contains the principal congruence subgroup $\Gamma_g(N)$ of $Sp\,(g,\mathbf{Z})$ for some N ([BLS], [Me]). We may assume that $N\geq 3$. Then the action of $\Gamma_g(N)$ on $\mathfrak{S}_g$ is fixed point free. Hence $X_g(N):=\Gamma_g(N)\backslash\mathfrak{S}_g$ is a manifold. $X_g(N)$ is a quasi-projective algebraic variety and is a open subspace of a projective variety $\overline{X}_g(N)$ which is called the Satake compactification ([Sta]). $\overline{X}_g(N)$ has singularities along its "cusps": $\overline{X}_g(N)-X_g(N)$, if $g\geq 2$. Smooth compactification of $X_g(N)$ was constructed in [Ig] when $g=2,3$ and in [Nm] when $g=2,3,4$ and more generally in [AMRT]. When $g=2,3$, the compactifications in [Ig] and in [Nm] coincide with each other and we denote them by $\widetilde{X}_g(N)$. The divisor "at infinity" $D:=\widetilde{X}_g(N)-X_g(N)$ is a divisor with simple normal crossings. In the following, we restrict ourselves to the case when $g=2$.

Let μ be a holomorphic representation of $GL(2,\mathbf{C})$ into $GL(r,\mathbf{C})$. Let $Z\in\mathfrak{S}_2$, $v\in\mathbf{C}^r$ and $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_2(N)$. We put

$$M(Z,v)=(M\langle Z\rangle,\mu(CZ+D)v).$$

This defines an action of $\Gamma_2(N)$ on $\mathcal{V}_\mu:=\mathfrak{S}_2\times\mathbf{C}^r$. If $N\geq 3$, this action is fixed point free and the quotient space $\Gamma_2(N)\backslash\mathcal{V}_\mu$ has a structure of a vector bundle over $X_2(N)$ which we denote by V_μ. V_μ has a natural extension to a vector bundle on $\widetilde{X}_2(N)$ which we denote by $\widetilde{V}_\mu$ ([Mu]). A holomorphic section of V_μ on $X_2(N)$ has an extension to a holomorphic section of $\widetilde{V}_\mu$ on $\widetilde{X}_2(N)$. Let $\mathcal{O}(\widetilde{V}_\mu-D)$ be the sheaf of the germs of the sections of $\widetilde{V}_\mu$ which vanish along the divisor D. Then the space of Siegel cusp form $S_\mu(\Gamma_g(N))$ is canonically identified with the space of the global sections $\Gamma(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu - D))$ (Proposition 5.9, below). Let $[D]$ be the line bundle on $\widetilde{X}_2(N)$ associated with the divisor D. $\mathcal{O}(\widetilde{V}_\mu - D)$ is isomorphic to $\mathcal{O}(\widetilde{V}_\mu \otimes [D]^{\otimes(-1)})$.

Let $G(N) = \Gamma_2(1)/\Gamma_2(N)$ and $G_0(N) = \Gamma_0(N)/\Gamma_2(N)$. We apply the results of §1 to the action of $G(N)$ on the pair $(\widetilde{X}_2(N), \widetilde{V}_\mu \otimes [D]^{\otimes(-1)})$. In [T2] the fixed points sets of $G(N)$ were classified as $\Phi_1, \Phi_2, \ldots, \Phi_{25}$ and their isotropy groups and stabilizer groups were determined. ($\Phi_1, \Phi_2, \ldots, \Phi_{14}$ intersect the quotient space $X_2(N)$. These were classified by [Gt].) Among the terms in Corollary 1.7, the order of the stabilizer group $|N_{G(N)}(\Phi_\lambda)|$ were determined in [T2]. (Note that since we studied the action of the group $\Gamma_2(1)/\pm\Gamma_2(N)$ in [T2], the value of $|N_{G(N)}(\Phi_\lambda)|$ in this paper is the double of the value in [T2].) $\tau(\varphi, \Phi_\lambda)$ were calculated in [T2] (case of weight k) and in [T3] (vector valued case, see §3 below for the details).

Let $N = p$ be a prime number. Then the classification of the conjugacy classes of $G(p)$ which is isomorphic to the symplectic group over finite field $\mathbf{F}_p$ is well known. Hence what we have to do is only to classify the conjugacy classes of $G_0(p)$. $G_0(p)$ is isomorphic to the following subgroup of $Sp(2, \mathbf{F}_p)$:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbf{F}_p) \,\middle|\, C = O \right\}.$$

First we describe the case $p = 2$. We list the conjugacy classes of $Sp(2, \mathbf{F}_2)$ in the following

Proposition 2.1. *$Sp(2, \mathbf{F}_2)$ has 11 conjugacy classes which we denote by $E_1, E_2, \ldots, E_6$, F_1, F_2, G_1, G_2 and H. The characteristic polynomials of E_i $(i = 1, 2, \ldots, 6)$, F_i $(i = 1, 2)$, G_i $(i = 1, 2)$ and H are $x^4 + 1$, $x^4 + x^2 + 1$, $x^4 + x^3 + x + 1$ and $x^4 + x^3 + x^2 + x + 1$, respectively. The representatives and the orders of the centralizer groups of them are as follows:*

$$E_1 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 720 \qquad E_2 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 48$$

$$E_3 \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 16 \qquad E_4 \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 48$$

E_5	$\begin{pmatrix}1&0&1&1\\1&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$	8	E_6	$\begin{pmatrix}1&0&1&0\\1&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$	8
F_1	$\begin{pmatrix}0&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix}$	18	F_2	$\begin{pmatrix}0&1&0&0\\1&1&1&0\\0&0&1&1\\0&0&1&0\end{pmatrix}$	6
G_1	$\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{pmatrix}$	18	G_2	$\begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&1&0\\0&0&0&1\end{pmatrix}$	6
H	$\begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\\1&0&0&1\end{pmatrix}$	5			

Next we classify the conjugacy classes of $G_0(2)$. As we saw in Corollary 1.7, the ratio of the orders of the centralizer groups $|C_{G(N)}(g)|/|C_{G_0(N)}(g)|$ is important rather than $|C_{G_0(N)}(g)|$. Therefore in the following two propositions, we list the ratio of the orders of the centralizer groups.

Proposition 2.2. *The conjugacy classes G_1, G_2 and H have no elements in $G_0(2)$. The conjugacy classes E_1, E_2, E_5, E_6, F_1 and F_2 do not split in $G_0(2)$ and their ratios are* $15, 3, 1, 1, 3, 1$, *respectively. The conjugacy class E_3 splits to two conjugacy classes in $G_0(2)$ which we denote by E_{3a} and E_{3b} and their ratios are* 1 *and* 2, *respectively. The conjugacy class E_4 splits to two conjugacy classes in $G_0(2)$ which we denote by E_{4a} and E_{4b} and their ratios are* 1 *and* 6, *respectively. The representatives of $E_{3a}, E_{3b}, E_{4a}, E_{4b}$ are as follows:*

$$\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad \begin{pmatrix}0&1&1&0\\1&0&0&1\\0&0&0&1\\0&0&1&0\end{pmatrix}, \quad \begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Next we study the case of odd prime. Let p be an odd prime. An element of $G_0(p)$ is represented by the following form:

$$\begin{pmatrix} A & S\,{}^tA^{-1} \\ O & {}^tA^{-1} \end{pmatrix},$$

where A belongs to $GL(2, \mathbf{F}_p)$ and S is a symmetric matrix in $M_2(\mathbf{F}_p)$. To save the space we denote this matrix by

$$(A \mid S).$$

We use the same notations as in [Sr] for the conjugacy classes of $G(p) = Sp\,(2, \mathbf{F}_p)$. If a conjugacy class of $G(p)$ does not split in $G_0(p)$, we denote the conjugacy class of $G_0(p)$ by the same notation. If a conjugacy class of $G(p)$ splits in $G_0(p)$ (for example, A_{31}), we denote the conjugacy classes by adding alphabet $a, b, \ldots$ to the suffix of the notation in [Sr] (for example, A_{31a}, A_{31b}). In the following proposition, we list the notation of the conjugacy class of $G_0(p)$, its representative and the ratio of the orders of the centralizer groups, in this order. Let θ be the generator of $\mathbf{F}_{p^2}^{\times}$ and let $\eta = \theta^{p-1}$ and $\gamma = \theta^{p+1}$. In the following proposition, we take as the representatives of $B_{2a}(i), B_{2b}(i), B_6(i), B_7(i)$ elements in $Sp\,(2, \mathbf{F}_{p^2})$ instead of in $Sp\,(2, \mathbf{F}_p)$.

Proposition 2.3. *The conjugacy classes of $G_0(p)$ are classified as follows:*

$$A_1, A_1' \qquad \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad (p^2+1)(p+1)$$

$$A_{21}, A_{21}' \qquad \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad p+1$$

$$A_{22}, A_{22}' \qquad \left(\begin{array}{cc|cc} 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad p+1$$

$$A_{31a}, A_{31a}' \qquad \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right) \qquad 1$$

$$A_{32}, A_{32}' \qquad \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\gamma \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\gamma \end{array}\right) \qquad 1$$

$$A_{31b}, A_{31b}' \qquad \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad 2p$$

$$A_{41}, A_{41}' \qquad \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \qquad 1$$

$$A_{42}, A_{42}' \qquad \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \gamma \end{array}\right), -\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \gamma \end{array}\right) \qquad 1$$

$$B_{2a}(i) \qquad \left(\begin{array}{cc|cc} \theta^i & 0 & 0 & 0 \\ 0 & \theta^{pi} & 0 & 0 \end{array}\right) \qquad 1$$

$$B_{2b}(i) \qquad \left(\begin{array}{cc|cc} \theta^{-i} & 0 & 0 & 0 \\ 0 & \theta^{-pi} & 0 & 0 \end{array}\right) \qquad 1$$

$$B_{3a}(i,j) \qquad \left(\begin{array}{cc|cc} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^j & 0 & 0 \end{array}\right) \qquad 1$$

$$B_{3b}(i,j) \qquad \left(\begin{array}{cc|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & \gamma^j & 0 & 0 \end{array}\right) \qquad 1$$

$B_{3c}(i,j)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^{-j} & 0 & 0 \end{array}\right)$	1
$B_{3d}(i,j)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & \gamma^{-j} & 0 & 0 \end{array}\right)$	1
$B_6(i)$	$\left(\begin{array}{cc\|cc} \eta^i & 0 & 0 & 0 \\ 0 & \eta^{-i} & 0 & 0 \end{array}\right)$	$p+1$
$B_7(i)$	$\left(\begin{array}{cc\|cc} \eta^i & 0 & 0 & 1 \\ 0 & \eta^{-i} & 1 & 0 \end{array}\right)$	1
$B_{8a}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^{i} & 0 & 0 \end{array}\right)$	1
$B_{8b}(i)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & \gamma^{-i} & 0 & 0 \end{array}\right)$	1
$B_{8c}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^{-i} & 0 & 0 \end{array}\right)$	$p+1$
$B_{9a}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 1 & 0 & 0 \\ 0 & \gamma^{i} & 0 & 0 \end{array}\right)$	1
$B_{9b}(i)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 1 & 0 & 0 \\ 0 & \gamma^{-i} & 0 & 0 \end{array}\right)$	1
$B_{9c}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 1 \\ 0 & \gamma^{-i} & 1 & 0 \end{array}\right)$	1
$C_{3a}(i), C'_{3a}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$	$p+1$
$C_{3b}(i), C'_{3b}(i)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$	$p+1$
$C_{41a}(i), C'_{41a}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$	1
$C_{41b}(i), C'_{41b}(i)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$	1
$C_{42a}(i), C'_{42a}(i)$	$\left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^i & 0 & 0 & 0 \\ 0 & -1 & 0 & \gamma \end{array}\right)$	1
$C_{42b}(i), C'_{42b}(i)$	$\left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma \end{array}\right), \left(\begin{array}{cc\|cc} \gamma^{-i} & 0 & 0 & 0 \\ 0 & -1 & 0 & \gamma \end{array}\right)$	1
D_1	$\left(\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$	$(p+1)^2$

D_{21}, D_{22}	$\left(\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right), \left(\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \gamma \end{array}\right)$	$p{+}1$
D_{23}, D_{24}	$\left(\begin{array}{cc\|cc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \left(\begin{array}{cc\|cc} 1 & 0 & \gamma & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$	$p{+}1$
D_{31}	$\left(\begin{array}{cc\|cc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$	1
D_{32}, D_{33}	$\left(\begin{array}{cc\|cc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & \gamma \end{array}\right), \left(\begin{array}{cc\|cc} 1 & 0 & \gamma & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$	1
D_{34}	$\left(\begin{array}{cc\|cc} 1 & 0 & \gamma & 0 \\ 0 & -1 & 0 & \gamma \end{array}\right)$	1

For $B_{2a}(i)$ and $B_{2b}(i)$, we assume that $\theta^i, \theta^{pi}, \theta^{-i}, \theta^{-pi}$ are distinct. For $B_{3a}(i,j), \dots, B_{3d}(i,j)$, we assume that $1 \le i < j \le (p-3)/2$. For $B_6(i)$ and $B_7(i)$, we assume that $1 \le i \le (p-1)/2$. For $B_{8a}(i), \dots, C'_{42b}(i)$, we assume that $1 \le i \le (p-3)/2$.

In [T2], we classified the proper elements of the isotropy groups of the fixed points sets $\Phi_1, \dots, \Phi_{25}$ as $\varphi_1, \varphi_2, \dots, \varphi_{25}(6, r, s, t)$. These elements are in $G(N)$ $(N \ge 3)$. $S_\mu(\Gamma_0(2))$ is the invariant subspace of $S_\mu(\Gamma_2(2N))$ by the action of $\Gamma_0(2)/\Gamma_2(2N)$ $(N \ge 2)$. We assume that N is odd. Then we have

$$\Gamma_0(2)/\Gamma_2(2N) \simeq G_0(2) \times G(N) \subset G(2) \times G(N) \simeq \Gamma_2(1)/\Gamma_2(2N).$$

For an element φ of $\Gamma_0(2)/\Gamma_2(2N)$ which fixes points in $\widetilde{X}_2(2N)$, the ratio of the orders of its centralizer groups in $\Gamma_0(2)/\Gamma_2(2N)$ and in $\Gamma_2(1)/\Gamma_2(2N)$ is equal to the ratio of the orders of the centralizer groups in $G_0(2)$ and in $G(2)$ of φ mod 2. Hence the problem is reduced to determine the conjugacy class in $G(2)$ of φ mod 2. For odd prime p, $S_\mu(\Gamma_0(p))$ is the invariant subspace of $S_\mu(\Gamma_2(p))$ by the action of $G_0(p)$. Let φ be an element of $G_0(p)$ which fixes points in $\widetilde{X}_2(p)$. The problem is to determine the conjugacy class in $G(p)$ of φ. In the following proposition, we list φ, the conjugacy class of φ mod 2 in $G(2) = Sp\,(2, \mathbf{F}_2)$ and the conjugacy class of φ in $G(p) = Sp\,(2, \mathbf{F}_p)$ for the cases of $p > 2$, in this order. When the notation of the element of the isotropy groups includes some index (for example, "r" in $\varphi_{15}(r)$), this index belongs to $\mathbf{Z}/2N\mathbf{Z}$ if $p = 2$ and to $\mathbf{F}_p$ if $p > 2$. We list the condition that the element is proper (for example, "$r \ne 0$") in the case of $p > 2$ under the notation of the element.

To save the space, we list only one of φ_i and $-\varphi_i$. In fact we consider $\Gamma_2(1)/\pm\Gamma_2(N)$ here, because some elements satisfy $\varphi \equiv -\varphi$ and so if we consider $\Gamma_2(1)/\Gamma_2(N)$, the situation will be complicated. Sometimes we list only the representatives classified by the equivalence relation $\equiv$. Here $\equiv$ means the conjugacy relation among the elements of $C_G^p(\Phi_\lambda)/\pm 1$ in $N_G(\Phi_\lambda)/\pm 1$. In such cases, we mark the notation of the elements by *1 and list the value of $e(\varphi_i)$, at the end of the proposition. In case the notation of the conjugacy class includes some index (for example, "i" in $B_2(i)$), we do not give the explicit value of i, since the ratio of the centralizer groups does not depend on i.

Proposition 2.4. *The conjugacy classes to which $\varphi_1, \dots, \varphi_{25}(6, r, s, t)$ belong in $Sp\,(2, \mathbf{F}_2)$ and $Sp\,(2, \mathbf{F}_p)$ $(p > 2)$ are as follows. The conjugacy classes whose notation are enclosed by brackets have no elements in $G_0(2)$ or in $G_0(p)$.*

1)	φ_1	E_1	A_1
2)	φ_2	E_1	D_1
3)	φ_3	E_4	D_1
4)	φ_4	E_4	$\begin{cases} B_8(i), & if\ p \equiv 1 \pmod 4 \\ B_6(i), & if\ p \equiv 3 \pmod 4 \end{cases}$
5)	φ_5	E_3	$\begin{cases} B_8(i), & if\ p \equiv 1 \pmod 4 \\ B_6(i), & if\ p \equiv 3 \pmod 4 \end{cases}$
6)	$\varphi_6{}^{*1}$	F_1	$\begin{cases} A_{31}, & if\ p = 3 \\ B_8(i), & if\ p \equiv 1 \pmod 3 \\ B_6(i), & if\ p \equiv 2 \pmod 3 \end{cases}$
7)	$\varphi_7(1)$	E_2	$\begin{cases} C_3(i), & if\ p \equiv 1 \pmod 4 \\ (C_1(i)), & if\ p \equiv 3 \pmod 4 \end{cases}$
	$\varphi_7(2)$	E_2	*the same as above*
8)	$\varphi_8(1)$	(G_1)	$\begin{cases} A_{21}, & if\ p = 3 \\ C_3(i), & if\ p \equiv 1 \pmod 3 \\ (C_1(i)), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_8(2)$	(G_1)	*the same as above*
	$\varphi_8(3)$	(G_1)	$\begin{cases} D_{21}, & if\ p = 3 \\ C_3(i), & if\ p \equiv 1 \pmod 3 \\ (C_1(i)), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_8(4)$	(G_1)	*the same as above*
9)	$\varphi_9(1)$	E_3	$\begin{cases} B_8(i), & if\ p \equiv 1 \pmod 4 \\ B_6(i), & if\ p \equiv 3 \pmod 4 \end{cases}$

	$\varphi_9(2)^{*1}$	E_5	$\begin{cases} B_3(i,j), & if\ p \equiv 1 \pmod 8 \\ B_2(i), & if\ p \equiv \pm 3 \pmod 8 \\ (B_4(i,j)), & if\ p \equiv 7 \pmod 8 \end{cases}$
	$\varphi_9(3)^{*1}$	E_5	*the same as above*
10)	$\varphi_{10}(1)$	F_1	$\begin{cases} A_{32}, & if\ p = 3 \\ B_8(i), & if\ p \equiv 1 \pmod 3 \\ B_6(i), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_{10}(2)$	F_1	*the same as above*
	$\varphi_{10}(3)^{*1}$	F_1	$\begin{cases} D_{33}, & if\ p = 3 \\ B_3(i,j), & if\ p \equiv 1 \pmod 3 \\ (B_4(i,j)), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_{10}(4)^{*1}$	F_2	$\begin{cases} D_{34}, & if\ p = 3 \\ B_3(i,j), & if\ p \equiv 1 \pmod 3 \\ (B_4(i,j)), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_{10}(5)^{*1}$	F_1	$\begin{cases} D_{31}, & if\ p = 3 \\ B_3(i,j), & if\ p \equiv 1 \pmod 3 \\ (B_4(i,j)), & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_{10}(6)$	F_1	*the same as in the case of* $\varphi_{10}(4)$
	$\varphi_{10}(7)$	F_2	*the same as in the case of* $\varphi_{10}(5)$
	$\varphi_{10}(8)^{*1}$	F_2	$\begin{cases} B_8(i), & if\ p = 3 \\ B_3(i,j), & if\ p \equiv 1 \pmod{12} \\ B_2(i), & if\ p \equiv \pm 5 \pmod{12} \\ (B_4(i,j)), & if\ p \equiv 11 \pmod{12} \end{cases}$
	$\varphi_{10}(9)^{*1}$	F_2	*the same as above*
11)	$\varphi_{11}(1)$	(G_2)	$\begin{cases} (C_{42}(i)), & if\ p = 3 \\ B_3(i,j), & if\ p \equiv 1 \pmod{12} \\ (B_5(i,j)), & if\ p \equiv \pm 5 \pmod{12} \\ (B_4(i,j)), & if\ p \equiv 11 \pmod{12} \end{cases}$
	$\varphi_{11}(2)$	(G_2)	$\begin{cases} (C_{41}(i)), & if\ p = 3 \\ \text{the same as above} & \\ \text{for the other cases} & \end{cases}$
	$\varphi_{11}(3)$	(G_2)	*the same as in the case of* $\varphi_{11}(1)$
	$\varphi_{11}(4)$	(G_2)	*the same as in the case of* $\varphi_{11}(2)$

12)	${\varphi_{12}}^{*1}$	F_2	$\begin{cases} D_{32}, & if\ p=3 \\ B_3(i,j), & if\ p\equiv 1 \pmod 3 \\ (B_4(i,j)), & if\ p\equiv 2 \pmod 3 \end{cases}$
13)	${\varphi_{13}}^{*1}$	E_6	*the same as in the case of* $\varphi_9(2)$
14)	$\varphi_{14}(1)$	(H)	$\begin{cases} A_{41}, & if\ p=5 \\ B_3(i,j), & if\ p\equiv 1 \pmod 5 \\ (B_4(i,j)), & if\ p\equiv 4 \pmod 5 \\ (B_1(i)), & if\ p\equiv \pm 2 \pmod 5 \end{cases}$
	$\varphi_{14}(2)$	(H)	*the same as above*
	$\varphi_{14}(3)$	(H)	*the same as above*
	$\varphi_{14}(4)$	(H)	*the same as above*
15)	$\varphi_{15}(r)$ $r\neq 0$	$\begin{cases} E_1, & if\ r\ is\ even \\ E_2, & if\ r\ is\ odd \end{cases}$	A_{21} *or* ${A_{22}}^{*2}$
16)	$\varphi_{16}(r)$ $r\neq 0$	$\begin{cases} E_1, & if\ r\ is\ even \\ E_2, & if\ r\ is\ odd \end{cases}$	D_{23} *or* D_{24}
17)	$\varphi_{17}(r)$ $r\neq 0$	$\begin{cases} E_4, & if\ r\ is\ even \\ E_3, & if\ r\ is\ odd \end{cases}$	D_{23} *or* D_{24}
18)	$\varphi_{18}(1,r)$ $r\neq 0$	$\begin{cases} E_2, & if\ r\ is\ even \\ E_3, & if\ r\ is\ odd \end{cases}$	$\begin{cases} C_{41}(i)\ or\ C_{42}(i), & if\ 4\mid p-1 \\ (C_{21}(i))\ or\ (C_{22}(i)), & if\ 4\nmid p-1 \end{cases}$
	$\varphi_{18}(2,r)$ $r\neq 0$	*the same as above*	*the same as above*
19)	$\varphi_{19}(1,r)$	$\begin{cases} E_5, & if\ r\ is\ even \\ E_6, & if\ r\ is\ odd \end{cases}$	$\begin{cases} {*3}\ below, & if\ p\equiv 1 \pmod 4 \\ {*4}\ below, & if\ p\equiv 3 \pmod 4 \end{cases}$
	$\varphi_{19}(2,r)$	*the same as above*	*the same as above*
20)	$\varphi_{20}(1,r)$ $r\neq 0$	$\begin{cases} (G_1), & if\ r\ is\ even \\ (G_2), & if\ r\ is\ odd \end{cases}$	$\begin{cases} A_{32}\ or\ A_{31}, & if\ p=3 \\ C_{41}(i)\ or\ C_{42}(i), & if\ 3\mid p-1 \\ (C_{21}(i))\ or\ (C_{22}(i)), & otherwise \end{cases}$
	$\varphi_{20}(2,r)$ $r\neq 0$	$\begin{cases} (G_1), & if\ r\ is\ even \\ (G_2), & if\ r\ is\ odd \end{cases}$	$\begin{cases} A_{31}\ or\ A_{32}, & if\ p=3 \\ C_{41}(i)\ or\ C_{42}(i), & if\ 3\mid p-1 \\ (C_{21}(i))\ or\ (C_{22}(i)), & otherwise \end{cases}$
	$\varphi_{20}(3,r)$ $r\neq 0$	$\begin{cases} (G_1), & if\ r\ is\ even \\ (G_2), & if\ r\ is\ odd \end{cases}$	$\begin{cases} D_{31}\ or\ D_{33}, & if\ p=3 \\ C_{41}(i)\ or\ C_{42}(i), & if\ 3\mid p-1 \\ (C_{21}(i))\ or\ (C_{22}(i)), & otherwise \end{cases}$

	$\varphi_{20}(4,r)$ $r \neq 0$	$\begin{cases} (G_1), \ if\ r\ is\ even \\ (G_2), \ if\ r\ is\ odd \end{cases}$	$\begin{cases} D_{32}\ or\ D_{34}, & if\ p=3 \\ C_{41}(i)\ or\ C_{42}(i), & if\ 3 \mid p-1 \\ (C_{21}(i))\ or\ (C_{22}(i)), & otherwise \end{cases}$
21)	$\varphi_{21}(1,r)$	$\begin{cases} (G_1), \ if\ r\ is\ even \\ (G_2), \ if\ r\ is\ odd \end{cases}$	$\begin{cases} A_{41}, & if\ p=3 \\ {*3}\ below, & if\ p \equiv 1 \pmod 3 \\ {*4}\ below, & if\ p \equiv 2 \pmod 3 \end{cases}$
	$\varphi_{21}(2,r)$	$\begin{cases} (G_1), \ if\ r\ is\ even \\ (G_2), \ if\ r\ is\ odd \end{cases}$	$\begin{cases} A_{42}, & if\ p=3 \\ {*3}\ below, & if\ p \equiv 1 \pmod 3 \\ {*4}\ below, & if\ p \equiv 2 \pmod 3 \end{cases}$
22)	$\varphi_{22}(1,r,t)$ $rt \neq 0$	$\begin{cases} E_1, \ if\ r\ and\ t\ are\ even \\ E_2, \ if\ r+t\ is\ odd \\ E_3, \ if\ r\ and\ t\ are\ odd \end{cases}$	$\begin{cases} A_{31}, \ if\ \left(\frac{-rt}{p}\right)=1 \\ A_{32}, \ if\ \left(\frac{-rt}{p}\right)=-1 \end{cases}$
	$\varphi_{22}(3,r,t)$ $r+t \neq 0$	$\begin{cases} E_4, \ if\ r+t\ is\ even \\ E_5, \ if\ r+t\ is\ odd \end{cases}$	$\begin{cases} D_{31}, \ if\ \left(\frac{r+t}{p}\right)=1 \\ D_{34}, \ if\ \left(\frac{r+t}{p}\right)=-1 \end{cases}$
23)	$\varphi_{23}(2,r,t)$ $r+t \neq 0$	$\begin{cases} E_4, \ if\ r+t\ is\ even \\ E_5, \ if\ r+t\ is\ odd \end{cases}$	$\begin{cases} B_9(i), \ if\ p \equiv 1 \pmod 4 \\ B_7(i), \ if\ p \equiv 3 \pmod 4 \end{cases}$
	$\varphi_{23}(4,r,t)$ $rt \neq 0$	$\begin{cases} E_1, \ if\ r\ and\ t\ are\ even \\ E_2, \ if\ r+t\ is\ odd \\ E_3, \ if\ r\ and\ t\ are\ odd \end{cases}$	$\begin{cases} D_{31}, \ if\ \left(\frac{r}{p}\right)=\left(\frac{t}{p}\right)=1 \\ D_{32}, \ if\ -\left(\frac{r}{p}\right)=\left(\frac{t}{p}\right)=1 \\ D_{33}, \ if\ \left(\frac{r}{p}\right)=-\left(\frac{t}{p}\right)=1 \\ D_{34}, \ if\ \left(\frac{r}{p}\right)=\left(\frac{t}{p}\right)=-1 \end{cases}$
24)	$\varphi_{24}(2,r,t)$ $r+t \neq 0$	$\begin{cases} E_3, \ if\ r+t\ is\ even \\ E_6, \ if\ r+t\ is\ odd \end{cases}$	$\begin{cases} B_9(i), \ if\ p \equiv 1 \pmod 4 \\ B_7(i), \ if\ p \equiv 3 \pmod 4 \end{cases}$
	$\varphi_{24}(4,r,t)$ $rt \neq 0$	$\begin{cases} E_4, \ if\ r\ and\ t\ are\ even \\ E_3, \ if\ r+t\ is\ odd \\ E_2, \ if\ r\ and\ t\ are\ odd \end{cases}$	*the same as in the case of* $\varphi_{23}(4,r,t)$
25)	$\varphi_{25}(1,r,s,t)$ $s(r+s)(t+s) \neq 0$	$\begin{cases} E_1, & if\ r,\ s,\ t\ are\ even \\ E_3, & if\ exactly\ one\ of\ r,\ s,\ t\ is\ even \\ E_4, & if\ r\ and\ t\ are\ even\ and\ s\ is\ odd \\ E_2, & otherwise \end{cases}$	$\begin{cases} A_{31}, \ if\ \left(\frac{s^2-rt}{p}\right)=1 \\ A_{32}, \ if\ \left(\frac{s^2-rt}{p}\right)=-1 \\ A_{21}, \ if\ s^2=rt\ and\ \left(\frac{r}{p}\right)=1 \\ A_{22}, \ if\ s^2=rt\ and\ \left(\frac{r}{p}\right)=-1 \end{cases}$
	$\varphi_{25}(2,r,s,t)$ $r+s+t \neq 0$	$\begin{cases} F_1, \ if\ r+s+t\ is\ even \\ F_2, \ if\ r+s+t\ is\ odd \end{cases}$	$\begin{cases} {*5}\ below, & if\ p=3 \\ B_9(i), & if\ p \equiv 1 \pmod 3 \\ B_7(i), & if\ p \equiv 2 \pmod 3 \end{cases}$

$$\varphi_{25}(4,r,s,t) \quad (r+2s+t)s \neq 0 \quad \begin{cases} E_4, & \textit{if } s \textit{ is even and } r+t \textit{ is even} \\ E_5, & \textit{if } s \textit{ is even and } r+t \textit{ is odd} \\ E_3, & \textit{if } s \textit{ is odd and } r+t \textit{ is even} \\ E_6, & \textit{if } s \textit{ is odd and } r+t \textit{ is odd} \end{cases} \quad \text{*6 } \textit{below}$$

*1 $e(\varphi_6)=2,\ e(\varphi_9(2))=2,\ e(\varphi_9(3))=2,\ e(\varphi_{10}(3))=2,\ e(\varphi_{10}(4))=3,\ e(\varphi_{10}(5))=3,$ $e(\varphi_{10}(8))=3,\ e(\varphi_{10}(9))=3,\ e(\varphi_{12})=2,\ e(\varphi_{13})=6.$

*2 *For 15), "A_{21} or A_{22}" means "A_{21}, if $\left(\frac{r}{p}\right)=1$ and A_{22}, if $\left(\frac{r}{p}\right)=-1$". The same applies to 16), 17), 18) and 20).*

*3 $C_3(i)$, *if* $r=a$, $C_{41}(i)$, *if* $\left(\frac{r-a}{p}\right)=1$, $C_{42}(i)$, *otherwise, where a is* $(p+1)/2$, $(p-1)/2$, $(1-p)/3$, $(p-1)/3$, *for* $\varphi_{19}(1,r)$, $\varphi_{19}(2,r)$, $\varphi_{21}(1,r)$, $\varphi_{21}(2,r)$, *respectively.*

*4 $(C_1(i))$, *if* $r=a$, $(C_{21}(i))$, *if* $\left(\frac{r-a}{p}\right)=1$, $(C_{22}(i))$, *otherwise, where a is* $(p+1)/2$, $(p-1)/2$, $(p+1)/3$, $-(p+1)/3$, *for* $\varphi_{19}(1,r)$, $\varphi_{19}(2,r)$, $\varphi_{21}(1,r)$, $\varphi_{21}(2,r)$, *respectively.*

*5 A_{41}, *if* $\left(\frac{r+s+t}{3}\right)=1$, A_{42}, *if* $\left(\frac{r+s+t}{3}\right)=-1$.

*6 D_{31}, *if* $\left(\frac{r+2s+t}{p}\right)=\left(\frac{r-2s+t}{p}\right)=1$, D_{32}, *if* $\left(\frac{r+2s+t}{p}\right)=-\left(\frac{r-2s+t}{p}\right)=1$, D_{33}, *if* $-\left(\frac{r+2s+t}{p}\right)=\left(\frac{r-2s+t}{p}\right)=1$, D_{34}, *if* $\left(\frac{r+2s+t}{p}\right)=\left(\frac{r-2s+t}{p}\right)=-1$, D_{23}, *if* $\left(\frac{r+2s+t}{p}\right)=1$ *and* $r+t=2s$, D_{24}, *if* $\left(\frac{r+2s+t}{p}\right)=-1$ *and* $r+t=2s$.

Remark 2.5. In the above theorem we omitted $\varphi_{25}(3,r,s,t)$, $\varphi_{25}(5,r,s,t)$ and $\varphi_{25}(6,r,s,t)$, because $\varphi_{25}(3,r,s,t)$ is equivalent by the relation "$\equiv$" to $\varphi_{25}(2,r',s',t')$ and $\varphi_{25}(5,r,s,t)$ and $\varphi_{25}(6,r,s,t)$ are equivalent to $\varphi_{25}(4,r',s',t')$. Instead of them, it suffices to double (resp. treble) the contribution of $\varphi_{25}(2,r,s,t)$ (resp. $\varphi_{25}(4,r,s,t)$) in the dimension formula (Corollary 1.7).

§3. Details of $|N_G(\Phi_\lambda)|$'s and $\tau(\varphi,\Phi_\lambda)$'s

Let μ be an irreducible holomorphic representation of $GL(2,\mathbf{C})$ and $(j+k,k)$ its signature. Then μ is equivalent to $s_j\otimes\det^k$, where s_j is the symmetric tensor representation of degree j and det is the alternating tensor representation of degree two of $GL(2,\mathbf{C})$, respectively. Let $N\geq 3$. As in §2, we consider the action of $G(N)$ on the pair $(\widetilde{X}_2(N),\widetilde{V}_\mu\otimes[D]^{\otimes(-1)})$. In this section we list the orders of $N_{G(N)}(\Phi_1), N_{G(N)}(\Phi_2),\ldots,N_{G(N)}(\Phi_{25})$ and the values of $\tau(\varphi_1,\Phi_1),\tau(\varphi_2,\Phi_2),\ldots,\tau(\varphi_{25}(4,r,s,t),\Phi_{25})$. The following theorem was obtained in [T2]. In the theorem, $\prod$ means $\prod_{p|N,\ p:prime}$.

Theorem 3.1. *The orders of the stabilizer groups of* $\Phi_1, \Phi_2, \dots, \Phi_{25}$ *are as follows:*

1) $|N_{G(N)}(\Phi_1)| = N^{10}\prod(1-p^{-2})(1-p^{-4})$

2) $|N_{G(N)}(\Phi_2)| = 2N^6\prod(1-p^{-2})^2$

3) $|N_{G(N)}(\Phi_3)| = \begin{cases} 2N^6\prod(1-p^{-2})^2, & if\ 2\nmid N \\ (8/3)N^6\prod(1-p^{-2})^2, & if\ 2\mid N \end{cases}$

4) $|N_{G(N)}(\Phi_4)| = 4N^3\prod(1-p^{-2})$

5) $|N_{G(N)}(\Phi_5)| = \begin{cases} 8N^3\prod(1-p^{-2}), & if\ 2\nmid N \\ (16/3)N^3\prod(1-p^{-2}), & if\ 2\mid N \end{cases}$

6) $|N_{G(N)}(\Phi_6)| = \begin{cases} 12N^3\prod(1-p^{-2}), & if\ 3\nmid N \\ 9N^3\prod(1-p^{-2}), & if\ 3\mid N \end{cases}$

7) $|N_{G(N)}(\Phi_7)| = 4N^3\prod(1-p^{-2})$

8) $|N_{G(N)}(\Phi_8)| = 6N^3\prod(1-p^{-2})$

9) $|N_{G(N)}(\Phi_9)| = 32$

10) $|N_{G(N)}(\Phi_{10})| = 72$

11) $|N_{G(N)}(\Phi_{11})| = 24$

12) $|N_{G(N)}(\Phi_{12})| = 24$

13) $|N_{G(N)}(\Phi_{13})| = 48$

14) $|N_{G(N)}(\Phi_{14})| = 10$

15) $|N_{G(N)}(\Phi_{15})| = 2N^6\prod(1-p^{-2})$

16) $|N_{G(N)}(\Phi_{16})| = 2N^4\prod(1-p^{-2})$

17) $|N_{G(N)}(\Phi_{17})| = \begin{cases} 2N^4\prod(1-p^{-2}), & if\ 2\nmid N \\ (8/3)N^4\prod(1-p^{-2}), & if\ 2\mid N \end{cases}$

18) $|N_{G(N)}(\Phi_{18})| = 8N$

19) $|N_{G(N)}(\Phi_{19})| = 8N$

20) $|N_{G(N)}(\Phi_{20})| = 12N$

21) $|N_{G(N)}(\Phi_{21})| = 6N$

22) $|N_{G(N)}(\Phi_{22})| = 8N^3$

23) $|N_{G(N)}(\Phi_{23})| = 8N^2$

24) $|N_{G(N)}(\Phi_{24})| = 8N^2$

25) $|N_{G(N)}(\Phi_{25})| = 12N^3$

We list the values of $\tau(\varphi_1, \Phi_1), \tau(\varphi_2, \Phi_2), \ldots, \tau(\varphi_{25}(4, r, s, t), \Phi_{25})$ in the following theorem. In the theorem ρ, ω and σ mean $\exp(2\pi i/3)$, $\exp(2\pi i/5)$ and $\exp(\pi i/6)$, respectively and $\mathrm{Tr}_\alpha(a)$ means $\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(a)$ for an algebraic number α and $a \in \mathbf{Q}(\alpha)$. Here we assume that j is even and replace j with $2j$. So the signature of μ is $(2j+k, k)$. We omit all of the details of the calculation. 1) is proved in §5 (Theorem 5.10).

Theorem 3.2. [4] $\tau(\varphi_1, \Phi_1), \tau(\varphi_2, \Phi_2), \ldots, \tau(\varphi_{25}(4, r, s, t), \Phi_{25})$ *are as follows (cf. Remark 2.5):*

1) $$\tau(\varphi_1, \Phi_1) = 2^{-8}3^{-3}5^{-1}((2j+1)(k-2)(2j+k-1)(2j+2k-3)N^{10} - 60(2j+1)(2j+2k-3)N^8 + 360(2j+1)N^7)\prod(1-p^{-2})(1-p^{-4})$$

2) $$\tau(\varphi_2, \Phi_2) = 2^{-7}3^{-3}(-1)^k((k-2)(2j+k-1)N^6 - 6(2j+2k-3)N^5 + 36N^4)\prod(1-p^{-2})^2$$

3) $$\tau(\varphi_3, \Phi_3) = (-1)^k((k-2)(2j+k-1)N^6 - 3(2j+2k-3)N^5 + 12N^4) \times \begin{cases} 2^{-6}3^{-1}\prod(1-p^{-2})^2, & if\ 2 \nmid N \\ 2^{-4}3^{-2}\prod(1-p^{-2})^2, & if\ 2 \mid N \end{cases}$$

4) $$\tau(\varphi_4, \Phi_4) = 2^{-5}3^{-1}(-1)^j((2j+2k-3)N^3 - 12N^2)\prod(1-p^{-2})$$

5) $$\tau(\varphi_5, \Phi_5) = (-1)^j((2j+2k-3)N^3 - 8N^2) \times \begin{cases} 2^{-5}\prod(1-p^{-2}), & if\ 2 \nmid N \\ 2^{-4}3^{-1}\prod(1-p^{-2}), & if\ 2 \mid N \end{cases}$$

6) $$\tau(\varphi_6, \Phi_6) = \mathrm{Tr}_\rho(\rho^j(1-\rho))((2j+2k-3)N^3 - 9N^2) \times \begin{cases} 2^{-1}3^{-3}\prod(1-p^{-2}), & if\ 3 \nmid N \\ 2^{-3}3^{-2}\prod(1-p^{-2}), & if\ 3 \mid N \end{cases}$$

[4]There are misprints in 2), 16) and 17). See the end of this paper.

7) $\tau(\varphi_7(1),\Phi_7) = 2^{-5}3^{-1}(i)^k(-1)^j((k-2)N^3-6N^2)\prod(1-p^{-2})$

$$+\,2^{-5}3^{-1}(i)^k(i)((2j+k-1)N^3-6N^2)\prod(1-p^{-2})$$

$\tau(\varphi_7(2),\Phi_7) =$ *the conjugate of* $\tau(\varphi_7(1),\Phi_7)$ *over* $\mathbf{Q}$

8) $\tau(\varphi_8(1),\Phi_8) = 2^{-3}3^{-3}(\rho^2)^k(\rho)^j(1-\rho^2)((k-2)N^3-6N^2)\prod(1-p^{-2})$

$$+\,2^{-3}3^{-3}(\rho^2)^k(1-\rho)((2j+k-1)N^3-6N^2)\prod(1-p^{-2})$$

$\tau(\varphi_8(2),\Phi_8) =$ *the conjugate of* $\tau(\varphi_8(1),\Phi_8)$ *over* $\mathbf{Q}$

$\tau(\varphi_8(3),\Phi_8) = 2^{-3}3^{-2}(-\rho^2)^k(\rho)^j(1-\rho^2)((k-2)N^3-6N^2)\prod(1-p^{-2})$

$$+\,2^{-3}3^{-2}(-\rho^2)^k(\rho-1)((2j+k-1)N^3-6N^2)\prod(1-p^{-2})$$

$\tau(\varphi_8(4),\Phi_8) =$ *the conjugate of* $\tau(\varphi_8(3),\Phi_8)$ *over* $\mathbf{Q}$

9) $\tau(\varphi_9(1),\Phi_9) = 2^{-3}(-1)^k(-1)^j(2j+1)$

$\tau(\varphi_9(2),\Phi_9) = 2^{-2}(-i)^k(i)^j(1+i)$

$\tau(\varphi_9(3),\Phi_9) = 2^{-2}(i)^k(-i)^j(1-i)$

10) $\tau(\varphi_{10}(1),\Phi_{10}) = 3^{-2}(\rho)^k(\rho)^j(2\rho+1)(2j+1)$

$\tau(\varphi_{10}(2),\Phi_{10}) = 3^{-2}(\rho^2)^k(\rho^2)^j(2\rho^2+1)(2j+1)$

$\tau(\varphi_{10}(3),\Phi_{10}) = 2^{-1}3^{-1}(-1)^k\mathrm{Tr}_\rho((\rho)^j(-\rho^2))$

$\tau(\varphi_{10}(4),\Phi_{10}) = \tau(\varphi_{10}(7),\Phi_{10}) = 3^{-1}(-\rho)^k(\rho)^j$

$\tau(\varphi_{10}(5),\Phi_{10}) = \tau(\varphi_{10}(6),\Phi_{10}) = 3^{-1}(-\rho^2)^k(\rho^2)^j$

$\tau(\varphi_{10}(8),\Phi_{10}) = 3^{-1}(\rho^2)^k(-\rho^2)^j(1+2\rho)$

$\tau(\varphi_{10}(9),\Phi_{10}) = 3^{-1}(\rho)^k(-\rho)^j(1+2\rho^2)$

11) $\tau(\varphi_{11}(1),\Phi_{11}) = 2^{-1}3^{-1}(\sigma^7)^k(-1)^j(1+\sigma^2) - 2^{-1}3^{-1}(\sigma^7)^k(\sigma^8)^j(\sigma+\sigma^3)$

$\tau(\varphi_{11}(2),\Phi_{11}) = 2^{-1}3^{-1}(\sigma^5)^k(-1)^j(1+\sigma^{10}) - 2^{-1}3^{-1}(\sigma^5)^k(\sigma^4)^j(\sigma^{11}+\sigma^9)$

$\tau(\varphi_{11}(3),\Phi_{11}) = 2^{-1}3^{-1}(\sigma)^k(-1)^j(1+\sigma^2) - 2^{-1}3^{-1}(\sigma)^k(\sigma^8)^j(\sigma^7+\sigma^9)$

$$\tau(\varphi_{11}(4),\Phi_{11}) = 2^{-1}3^{-1}(\sigma^{11})^k(-1)^j(1+\sigma^{10}) - 2^{-1}3^{-1}(\sigma^{11})^k(\sigma^4)^j(\sigma^5+\sigma^3)$$

12) $\tau(\varphi_{12},\Phi_{12}) = 2^{-1}3^{-1}(-1)^k\mathrm{Tr}_\rho((\rho)^j(-\rho^2))$

13) $\tau(\varphi_{13},\Phi_{13}) = 2^{-3}(-1)^k\mathrm{Tr}_i((i)^j(1+i))$

14) $\tau(\varphi_{14}(1),\Phi_{14}) = 5^{-1}(\omega)^k(\omega^4)^j - 5^{-1}(\omega)^k(\omega^3)^j\omega^2$

$$\tau(\varphi_{14}(2),\Phi_{14}) = 5^{-1}(\omega^2)^k(\omega^3)^j - 5^{-1}(\omega^2)^k(\omega)^j\omega^4$$

$$\tau(\varphi_{14}(3),\Phi_{14}) = 5^{-1}(\omega^3)^k(\omega^2)^j - 5^{-1}(\omega^3)^k(\omega^4)^j\omega$$

$$\tau(\varphi_{14}(4),\Phi_{14}) = 5^{-1}(\omega^4)^k(\omega)^j - 5^{-1}(\omega^4)^k(\omega^2)^j\omega^3$$

15) $$\tau(\varphi_{15}(r),\Phi_{15}) = 2^{-3}3^{-1}(2j+1)N^3\prod(1-p^{-2}) \times\left(\frac{9-(2j+2k-3)N}{(1-\zeta^r)} + \frac{(2j+2k-3)N-6}{(1-\zeta^r)^2} - \frac{4}{(1-\zeta^r)^3}\right)$$

16) $$\tau(\varphi_{16}(r),\Phi_{16}) = 2^{-5}3^{-1}(-1)^k\left(\frac{12-(2j+2k-3)N}{(1-\zeta^r)}\right)N^3\prod(1-p^{-2})$$

17) $$\tau(\varphi_{17}(r),\Phi_{17}) = (-1)^k\left(\frac{8-(2j+2k-3)N}{(1-\zeta^r)} + \frac{4}{(1-\zeta^r)^2}\right)N^3\prod(1-p^{-2}) \times\begin{cases} 2^{-5}, & if\ 2\nmid N \\ 2^{-3}3^{-1}, & if\ 2\mid N\end{cases}$$

18) $\tau(\varphi_{18}(1,r),\Phi_{18}) = 2^{-2}(-i)^k((-1)^j - i)(\zeta^r-1)^{-1}$

$$\tau(\varphi_{18}(2,r),\Phi_{18}) = 2^{-2}(i)^k((-1)^j + i)(\zeta^r-1)^{-1}$$

19) $\tau(\varphi_{19}(1,r),\Phi_{19}) = 2^{-2}(-i)^k((-1)^j - i)(\exp(\pi i(2r-1)/N)-1)^{-1}$

$$\tau(\varphi_{19}(2,r),\Phi_{19}) = 2^{-2}(i)^k((-1)^j + i)(\exp(\pi i(2r+1)/N)-1)^{-1}$$

20) $\tau(\varphi_{20}(1,r),\Phi_{20}) = 3^{-2}(\rho^2)^k(\rho^2(\rho)^j-1)(\rho-1)(\zeta^r-1)^{-1}$

$$\tau(\varphi_{20}(2,r),\Phi_{20}) = 3^{-2}(\rho)^k(\rho(\rho^2)^j-1)(\rho^2-1)(\zeta^r-1)^{-1}$$

$$\tau(\varphi_{20}(3,r),\Phi_{20}) = 3^{-1}(-\rho^2)^k(1+\rho^2(\rho)^j)(\rho-1)(\zeta^r-1)^{-1}$$

$$\tau(\varphi_{20}(4,r),\Phi_{20}) = 3^{-1}(-\rho)^k(1+\rho(\rho^2)^j)(\rho^2-1)(\zeta^r-1)^{-1}$$

21) $\tau(\varphi_{21}(1,r),\Phi_{21}) = 3^{-2}(\rho^2)^k(\rho^2(\rho)^j-1)(\rho-1)(\exp(2\pi i(3r-1)/3N)-1)^{-1}$

$$\tau(\varphi_{21}(2,r),\Phi_{21}) = 3^{-2}(\rho)^k(\rho(\rho^2)^j-1)(\rho^2-1)(\exp(2\pi i(3r+1)/3N)-1)^{-1}$$

22) $$\tau(\varphi_{22}(1,r,t),\Phi_{22}) = \frac{(2j+1)}{(\zeta^r-1)(\zeta^t-1)}\left(\frac{2}{(\zeta^r-1)}+\frac{2}{(\zeta^t-1)}+3\right)$$

$$\tau(\varphi_{22}(3,r,t),\Phi_{22}) = \frac{(-1)^k}{(\zeta^{r+t}-1)}\left(\frac{4}{(\zeta^{r+t}-1)}+3\right)$$

23) $$\tau(\varphi_{23}(2,r,t),\Phi_{23}) = 2^{-1}(-1)^j(\zeta^{r+t}-1)^{-1}$$

$$\tau(\varphi_{23}(4,r,t),\Phi_{23}) = 2^{-1}(-1)^k(\zeta^r-1)^{-1}(\zeta^t-1)^{-1}$$

24) $$\tau(\varphi_{24}(2,r,t),\Phi_{24}) = 2^{-1}(-1)^j(\zeta^{r+t}-1)^{-1}$$

$$\tau(\varphi_{24}(4,r,t),\Phi_{24}) = 2^{-1}(-1)^k(\zeta^r-1)^{-1}(\zeta^t-1)^{-1}$$

25) $$\tau(\varphi_{25}(1,r,s,t),\Phi_{25}) = (2j+1)(\zeta^{r+s}-1)^{-1}(\zeta^{s+t}-1)^{-1}(\zeta^{-s}-1)^{-1}$$

$$\tau(\varphi_{25}(2,r,s,t),\Phi_{25}) = 3^{-1}\mathrm{Tr}_\rho(\rho^j(1-\rho))(\zeta^{s+r+t}-1)^{-1}$$

$$\tau(\varphi_{25}(4,r,s,t),\Phi_{25}) = (-1)^k(\zeta^{r+2s+t}-1)^{-1}(\zeta^{-s}-1)^{-1}$$

§4. Dimension Formula and Exponential Sums

Let p be a prime number and χ a Dirichlet character modulo p. Let N be a natural number which is relatively prime to p. $S_\mu(\Gamma_0(p),\chi)$ is the invariant subspace of $S_\mu(\Gamma_2(pN))$ of the action of $G_0 := \Gamma_0(p)/\Gamma_2(pN)$ which is twisted by χ. By applying the results in §2, §3 and the vanishing theorem (Theorem 6.1) to the general dimension formula in §1, we can calculate the dimension of $S_\mu(\Gamma_0(p),\chi)$. We may assume $N=1$ if $p>2$ and N (≥ 3) is odd if $p=2$. Let $(j+k,k)$ be the signature of μ. Since -1_4 belongs to $\Gamma_0(p)$, $S_\mu(\Gamma_0(p),\chi)\simeq 0$ if j is odd. So in this section we assume the signature of μ is $(2j+k,k)$.

Let p be an odd prime. When we calculate the contributions of the fixed points at infinity, we have to evaluate some exponential sums. Almost all of them are evaluated easily and we omit here the evaluation of them. But in the following cases, the evaluations are rather complicated and in a certain case, we can not evaluate it directly.

As we saw in §2, the conjugacy class in $G(p)$ of $\varphi_{22}(1,r,t)$ is A_{31} if $\left(\frac{-rt}{p}\right)=1$ and A_{32} if $\left(\frac{-rt}{p}\right)=-1$. In $G_0(p)$, A_{31} splits to A_{31a} and A_{31b}. But A_{32} does not split. Therefore by the results in §1 and in §2, the contribution of $\varphi_{22}(1,r,t)$'s to $\dim S_\mu(\Gamma_0(p),\chi)$ is equal to

$$\frac{2}{8p^3}\left\{(2p+1)\sum{}' \tau(\varphi_{22}(1,r,t),\Phi_{22}) + \sum{}'' \tau(\varphi_{22}(1,r,t),\Phi_{22})\right\},$$

where in the first sum, (r,t) is over the pairs such that $\left(\frac{-rt}{p}\right)=1$ and in the second sum, (r,t) is over the pairs such that $\left(\frac{-rt}{p}\right)=-1$. The above sums are rewritten as

$$\begin{aligned}&\frac{1}{4p^3}\left\{(p+1)\sum_{r,t=1}^{p-1}\tau(\varphi_{22}(1,r,t),\Phi_{22})+p\sum_{r,t=1}^{p-1}\left(\frac{-rt}{p}\right)\tau(\varphi_{22}(1,r,t),\Phi_{22})\right\}\\ =&\frac{1}{4p^3}\left\{\frac{(2j+1)(p+1)(p-1)^2(2p-1)}{12}+p\sum_{r,t=1}^{p-1}\left(\frac{-rt}{p}\right)\tau(\varphi_{22}(1,r,t),\Phi_{22})\right\}.\end{aligned}$$

Therefore the problem is reduced to evaluate the sum in the last expression. To evaluate this sum, we recall the well-known formula of the Gaussian sums and the class numbers. We denote by $h(-p)$ the class number of $\mathbf{Q}(\sqrt{-p})$.

Theorem 4.1. *Let p be an odd prime and we denote $\exp(2\pi\sqrt{-1}/p)$ by ζ. Then*

$$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\zeta^k=\sqrt{\epsilon_p p},\ \textit{where}\ \epsilon_p=\left(\frac{-1}{p}\right).$$

$$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)k=\begin{cases}0, & \textit{if}\ p\equiv 1 \pmod 4\\ -ph(-p)/\omega_p, & \textit{if}\ p\equiv 3 \pmod 4\end{cases},$$

where $\omega_3=3$ and $\omega_p=1$, otherwise.

Corollary 4.2.

(a) $$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{1}{1-\zeta^k}=\begin{cases}0, & \textit{if}\ p\equiv 1 \pmod 4\\ \sqrt{-p}h(-p)/\omega_p, & \textit{if}\ p\equiv 3 \pmod 4\end{cases}.$$

(b) $$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{1}{(1-\zeta^k)^2}=\sqrt{-p}h(-p)/\omega_p,\ \textit{if}\ p\equiv 3 \pmod 4.$$

Proof. Since

$$\frac{1}{1-\zeta^k}=-\frac{1}{p}\sum_{r=1}^{p-1}r\zeta^{kr},$$

we have

$$\begin{aligned}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{1}{1-\zeta^k}&=-\frac{1}{p}\sum_{k,r=1}^{p-1}\left(\frac{k}{p}\right)r\zeta^{kr}\\ &=-\frac{1}{p}\sum_{k,r=1}^{p-1}\left(\frac{r}{p}\right)r\left(\frac{kr}{p}\right)\zeta^{kr}\\ &=-\frac{1}{p}\sum_{r=1}^{p-1}\left(\frac{r}{p}\right)r\sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\zeta^{s}.\end{aligned}$$

Hence (a) is proved by Theorem 3.1. (b) is proved from

$$\frac{1}{(1-\zeta^k)^2} = \frac{1}{2p}\sum_{r=1}^{p-1}((p-2)r - r^2)\zeta^{kr}$$

and

$$\begin{aligned}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)k^2 &= \frac{1}{2}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)(k^2-(p-k)^2) \\ &= p\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)k.\end{aligned}$$

We put

$$X_p = \sum_{r,t=1}^{p-1}\left(\frac{-rt}{p}\right)\frac{1}{(\zeta^r-1)(\zeta^t-1)}\left(\frac{2}{(\zeta^r-1)}+\frac{2}{(\zeta^t-1)}+3\right).$$

Then by Corollary 4.2, we have

$$X_p = \begin{cases} 0, & if\ p\equiv 1 \pmod 4 \\ -ph(-p)^2/\omega_p^2, & if\ p\equiv 3 \pmod 4\end{cases}.$$

and

Proposition 4.3. *The contribution of $\varphi_{22}(1,r,t)$'s is*

$$\frac{(2j+1)(p+1)(p-1)^2(2p-1)}{48p^3} - \frac{(2j+1)}{4p^2}\begin{cases} 0, & if\ p\equiv 1 \pmod 4 \\ ph(-p)^2/\omega_p^2, & if\ p\equiv 3 \pmod 4\end{cases}.$$

Next we consider the contribution of $\varphi_{25}(1,r,s,t)$'s. Similarly as above by using the results in §2, the contribution is expressed as

$$\frac{2}{12p^3}\Bigg\{(2p+1){\sum}'\tau(\varphi_{25}(1,r,s,t),\Phi_{25}) + {\sum}''\tau(\varphi_{25}(1,r,s,t),\Phi_{25}) + p{\sum}'''\tau(\varphi_{25}(1,r,s,t),\Phi_{25})\Bigg\},$$

where in the first sum, (r,s,t) is over the triples such that $\left(\frac{s^2-rt}{p}\right)=1$, in the second sum, (r,s,t) is over the triples such that $\left(\frac{s^2-rt}{p}\right)=-1$ and in the third sum, (r,s,t) is over the triples such that $s^2 = rt$. In each sum, it is assumed that $s(r+s)(s+t) \neq 0$. The above sums are rewritten as

$$\frac{1}{6p^3}\Bigg\{(p+1)\sum \tau(\varphi_{25}(1,r,s,t),\Phi_{25}) + p\sum_{s^2\neq rt}\left(\frac{s^2-rt}{p}\right)\tau(\varphi_{25}(1,r,s,t),\Phi_{25}) - \sum_{s^2=rt}\tau(\varphi_{25}(1,r,s,t),\Phi_{25})\Bigg\}$$
$$=\frac{(2j+1)}{6p^3}\left(-\frac{(p+1)(p-1)^3}{8}+pY_p - Z_p\right),$$

where we put

$$Y_p = \sum_{s^2\neq rt}\left(\frac{s^2-rt}{p}\right)(\zeta^{r+s}-1)^{-1}(\zeta^{s+t}-1)^{-1}(\zeta^{-s}-1)^{-1},$$
$$Z_p = \sum_{s^2=rt}(\zeta^{r+s}-1)^{-1}(\zeta^{s+t}-1)^{-1}(\zeta^{-s}-1)^{-1}.$$

We can prove

Proposition 4.4. $Z_p = 0$.

Proof. We put $k = r/s = s/t$. Then Z_p is rewritten as

$$Z_p = \sum_{k=1}^{p-2}\sum_{s=1}^{p-1}(\zeta^{(k+1)s}-1)^{-1}(\zeta^{(k^{-1}+1)s}-1)^{-1}(\zeta^{-s}-1)^{-1}.$$

We modify the above expression as follows.

$$\frac{1}{(1-\zeta^{(k+1)s})(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}$$
$$=\frac{(1-\zeta^{(k+1)s})+\zeta^{(k+1)s}}{(1-\zeta^{(k+1)s})(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}$$
$$=\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}+\frac{1}{(\zeta^{-(k+1)s}-1)(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}$$
$$=\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}+\frac{(1-\zeta^{(k^{-1}+1)s})+\zeta^{(k^{-1}+1)s}}{(\zeta^{-(k+1)s}-1)(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}$$
$$=\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}+\frac{1}{(\zeta^{-(k+1)s}-1)(1-\zeta^{-s})}+\frac{1}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)(1-\zeta^{-s})}$$
$$=\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}+\frac{1}{(\zeta^{-(k+1)s}-1)(1-\zeta^{-s})}$$

$$+\frac{(1-\zeta^{-s})+\zeta^{-s}}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)(1-\zeta^{-s})}$$
$$=\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}+\frac{1}{(\zeta^{-(k+1)s}-1)(1-\zeta^{-s})}$$
$$+\frac{1}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)}+\frac{1}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)(\zeta^{s}-1)}.$$

The sums of the first three terms in the last expression are evaluated as follows.

$$\sum_{k=1}^{p-2}\sum_{s=1}^{p-1}\frac{1}{(1-\zeta^{(k^{-1}+1)s})(1-\zeta^{-s})}=\sum_{m,n=1}^{p-1}\frac{1}{(1-\zeta^{m})(1-\zeta^{n})}-\sum_{s=1}^{p-1}\frac{1}{(1-\zeta^{s})(1-\zeta^{-s})}$$
$$=\frac{(p-1)^2}{4}-\frac{(p-1)(p+1)}{12}.$$

$$\sum_{k=1}^{p-2}\sum_{s=1}^{p-1}\frac{1}{(\zeta^{-(k+1)s}-1)(1-\zeta^{-s})}=\sum_{m,n=1}^{p-1}\frac{1}{(\zeta^{m}-1)(1-\zeta^{n})}-\sum_{s=1}^{p-1}\frac{1}{(\zeta^{-s}-1)(1-\zeta^{-s})}$$
$$=-\frac{(p-1)^2}{4}-\frac{(p-1)(p-5)}{12}.$$

In the third term, we put $u=(k^{-1}+1)s$. Then $(k+1)s=ku$. So we have

$$\sum_{k=1}^{p-2}\sum_{s=1}^{p-1}\frac{1}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)}=\sum_{k=1}^{p-2}\sum_{u=1}^{p-1}\frac{1}{(\zeta^{-ku}-1)(\zeta^{-u}-1)}$$
$$=\sum_{m,n=1}^{p-1}\frac{1}{(\zeta^{m}-1)(\zeta^{n}-1)}-\sum_{u=1}^{p-1}\frac{1}{(\zeta^{u}-1)(\zeta^{-u}-1)}$$
$$=\frac{(p-1)^2}{4}-\frac{(p-1)(p+1)}{12}.$$

The sum of these three sums is zero. Hence we have

$$-Z_p=\sum_{k=1}^{p-2}\sum_{s=1}^{p-1}\frac{1}{(\zeta^{-(k+1)s}-1)(\zeta^{-(k^{-1}+1)s}-1)(\zeta^{s}-1)}.$$

In the right-hand side, we can change s to $-s$. Therefore this is equal to Z_p. Thus we proved $Z_p=0$.

We have not evaluated Y_p. But we computed this sum by using computer for $p<500$ and obtained the following

Proposition 4.5. *For primes such that* $3\le p<500$ *we have*

$$Y_p=-\frac{p(p-1)^2}{8}+\begin{cases}0, & if\ p\equiv 1 \pmod 4\\ \frac{3}{2}ph(-p)^2/\omega_p^2, & if\ p\equiv 3 \pmod 4\end{cases}.$$

We evaluated all of the exponential sums which appear in the formula of $\dim S_\mu(\Gamma_0(p),\chi)$ except Y_p. Comparing our result with Hashimoto's result ([Ha1]) in the case of weight k and trivial χ, we derive the following

Theorem 4.6. *Proposition 4.5 holds for general odd primes.*

Hence we have

Proposition 4.7. *The contribution of* $\varphi_{25}(1,r,s,t)$*'s is*

$$-\frac{(2j+1)(p-1)^2(2p^2-1)}{48p^3}+\frac{(2j+1)}{4p^2}\times\begin{cases}0, & if\ p\equiv 1 \pmod 4\\ ph(-p)^2/\omega_p^2, & if\ p\equiv 3 \pmod 4\end{cases}.$$

Remark 4.8. The terms of the class number in X_p and in Y_p cancel with each other and do not appear in $\dim S_\mu(\Gamma_0(p),\chi)$.

Remark 4.9. Concerning the representation of $Sp\,(2,\mathbf{F}_p)$ on $S_k(\Gamma_2(p))$, a similar exponential sum which is represented as

$$\sum_{s^2=rt}\left(\frac{r}{p}\right)\tau(\phi_{25}(1,r,s,t),\Phi_{25})$$

by our notation was considered and a conjecture about this value was presented in [LW]. This conjecture was proved by [A2] and [IS].

Remark 4.10. Let $p>2$. We recall φ_1, $\varphi_{15}(r)$, $\varphi_{22}(1,r,t)$ and $\varphi_{25}(1,r,s,t)$. They are

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0&0\\0&1&0&r\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&r&0\\0&1&0&t\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}1&0&r&s\\0&1&s&t\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

respectively. We regard $\varphi_{22}(1,r,t)$ as the degenerated element of $\varphi_{25}(1,r,s,t)$. Similarly $\varphi_{15}(r)$ is the degenerated element of $\varphi_{22}(1,r,t)$ and φ_1 is the degenerated element of $\varphi_{15}(r)$. These elements constitute a series of degenerating elements. As we saw before, the contributions of φ_1, $\varphi_{15}(r)$'s, $\varphi_{22}(1,r,t)$'s and $\varphi_{25}(1,r,s,t)$'s include terms which are multiples of $\frac{1}{p}$, $\frac{1}{p^2}$ or $\frac{1}{p^3}$. For example the contribution of φ_1 is

$$\begin{aligned}&\frac{\tau(\varphi_1,\Phi_1)}{|N_{G(p)}|}\cdot\frac{|C_{G(p)}(\varphi_1)|}{|C_{G_0(p)}(\varphi_1)|}\\ =&2^{-8}3^{-3}5^{-1}(p+1)(p^2+1)\bigg((2j+1)(k-2)(2j+k-1)(2j+2k-3)\\ &\qquad-\frac{60(2j+1)(2j+2k-3)}{p^2}+\frac{360(2j+1)}{p^3}\bigg).\end{aligned}$$

But we know (experimentally) if we take the sum of the contributions of the elements in the series, then the terms which are multiples of $\frac{1}{p}$, $\frac{1}{p^2}$ or $\frac{1}{p^3}$ cancel with each other. Therefore in the following theorem, we list the sum of the contributions of each series.

Now by taking the sum of the contributions of all of the fixed points, we have

Theorem 4.11. *Let p be a prime number, h a generator of $(\mathbf{Z}/p\mathbf{Z})^\times$ and χ a Dirichlet character modulo p, and let $\zeta = \exp\left(\dfrac{2\pi i}{p-1}\right)$ and assume $\chi(h) = \zeta^u$ $(0 \le u \le p-2)$. Let N (≥ 3) be odd if $p = 2$ and $N = 1$ if $p > 2$ and let G_0 be $\Gamma_0(p)/\Gamma_2(pN)$. Let μ be the holomorphic representation of $GL(2,\mathbf{C})$ whose signature is $(2j+k,k)$. Then $\sum_i (-1)^i \dim H^i(\widetilde{X}_2(pN), \mathcal{O}(\widetilde{V}_\mu - D))^{G_0}$ is given by the following Mathematica function:*

```
Siegel[p_,u_,j_,k_]:=Block[{a,lj,lk,ljk,lu,x,y},

    mod[x_,y_]:=Mod[x,y]+1;

    a=(p+1)*(p^2+1)*(2*j+1)*(2*j+k-1)*(2*j+2*k-3)*(k-2)/5/27/128;
    a=a-(p+1)*(2*j+1)*(2*j+2*k-3)/9/16;
    a=a+(2*j+1)/12;
    (* contribution of φ1 *)
    (* contribution of φ15(r) *)
    (* contribution of φ22(1,r,t) *)
    (* contribution of φ25(1,r,s,t) *)

    lk={1,-1};
    lu={1,-1};
    a=a+If[p==2,15,(p+1)^2]*(2*j+k-1)*(k-2)/9/128*
          lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a-If[p==2,9,2*(p+1)]*(2*j+2*k-3)/3/64*lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a+If[p==2,3,2]*lk[[mod[k,2]]]*lu[[mod[u,2]]]/16;
    (* contribution of φ2 *)
    (* contribution of φ16(r) *)
    (* contribution of φ23(4,r,t) *)

    a=a+If[p==2,7,(p+1)^2]*(2*j+k-1)*(k-2)/3/64*
          lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a-If[p==2,5,2*(p+1)]*(2*j+2*k-3)/64*lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a+If[p==2,1/4,1/8]*lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a+1/8*If[Mod[p,4]==1,1,0]*lk[[mod[k,2]]]*lu[[mod[u,2]]];
    a=a+1/8*If[Mod[p,4]==3,2,0]*lk[[mod[k,2]]]*lu[[mod[u,2]]];
    (* contribution of φ3 *)
    (* contribution of φ17(r) *)
    (* contribution of φ22(3,r,t) *)
    (* contribution of φ24(4,r,t) *)
```

(* contribution of $\varphi_{25}(i,r,s,t)$ $(i=4,5,6)$ *)

```
lj={1,-1};
a=a+If[p==2,7,p+1]*(2*j+2*k-3)/3/64*lj[[mod[j,2]]];
a=a+If[Mod[p,4]==1,(2*j+2*k-3)/96*lj[[mod[j,2]]]*lu[[mod[u,2]]],0];
a=a-If[p==2,2,1]*lj[[mod[j,2]]]/8;
a=a-If[Mod[p,4]==1,lj[[mod[j,2]]]*lu[[mod[u,2]]],0]/8;
```

(* contribution of φ_4 *)

(* contribution of $\varphi_{23}(2,r,t)$ *)

```
a=a+(p+1)*(2*j+2*k-3)/128*lj[[mod[j,2]]];
a=a+If[Mod[p,4]==1,(2*j+2*k-3)/64*lj[[mod[j,2]]]*lu[[mod[u,2]]],0];
a=a-lj[[mod[j,2]]]/8;
a=a-If[Mod[p,4]==1,lj[[mod[j,2]]]*lu[[mod[u,2]]],0]/8;
```

(* contribution of φ_5 *)

(* contribution of $\varphi_{24}(2,r,t)$ *)

```
lj={1,0,-1};
lu={2,-1,-1};
a=a+If[p==3,7,p+1]*(2*j+2*k-3)*lj[[mod[j,3]]]/54;
a=a+If[Mod[p,3]==1,(2*j+2*k-3)*lj[[mod[j,3]]]*lu[[mod[u,3]]],0]/54;
a=a-If[p==3,1/2,1/3]*lj[[mod[j,3]]];
a=a-If[Mod[p,3]==1,lj[[mod[j,3]]]*lu[[mod[u,3]]],0]/6;
```

(* contribution of φ_6 *)

(* contribution of $\varphi_{25}(2,r,s,t)$ and $\varphi_{25}(3,r,s,t)$ *)

```
ljk={{-2+k,1-2*j-k,2-k,-1+2*j+k},{2-k,1-2*j-k,-2+k,-1+2*j+k}};
a=a+If[p==2,ljk[[mod[j,2],mod[k,4]]],0]/32;
lu={2,0,-2,0};
a=a+(p+1)*If[Mod[p,4]==1,ljk[[mod[j,2],mod[k,4]]],0]*
      lu[[mod[u,4]]]/96;
ljk={{-1,1,1,-1},{1,1,-1,-1}};
a=a+If[p==2,3*ljk[[mod[j,2],mod[k,4]]],0]/16;
a=a+If[Mod[p,4]==1,ljk[[mod[j,2],mod[k,4]]],0]*lu[[mod[u,4]]]/8;
```

(* contribution of $\varphi_7(1)$ and $\varphi_7(2)$ *)

(* contribution of $\varphi_{18}(1,r)$ and $\varphi_{18}(2,r)$ *)

```
a=a+If[p==2,ljk[[mod[j,2],mod[k,4]]],0]/16;
a=a+If[Mod[p,4]==1,ljk[[mod[j,2],mod[k,4]]],0]*lu[[mod[u,4]]]/8;
```

(* contribution of $\varphi_{19}(1,r)$ and $\varphi_{19}(2,r)$ *)

```
ljk={{-3+2*j+2*k,1-2*j-k,2-k},{1+2*j,-1-2*j,0},
      {-1+2*j+k,3-2*j-2*k,-2+k}};
lu={2,-1,-1};
a=a+(p+1)*If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,3]]],0]*
      lu[[mod[u,3]]]/216;
a=a+If[p==3,ljk[[mod[j,3],mod[k,3,3]]],0]/54;
ljk={{-2,1,1},{0,0,0},{-1,2,-1}};
a=a+If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,3]]],0]*lu[[mod[u,3]]]/18;
a=a+If[p==3,ljk[[mod[j,3],mod[k,3]]],0]/18;
ljk={{-1,1,0},{0,0,0},{0,1,-1}};
a=a+If[p==3,ljk[[mod[j,3],mod[k,3]]],0]/9;
```

(* contribution of $\varphi_8(1)$ and $\varphi_8(2)$ *)

(* contribution of $\varphi_{20}(1,r)$ and $\varphi_{20}(2,r)$ *)

```
ljk={{-1-2*j,1-2*j-k,2-k,1+2*j,-1+2*j+k,-2+k},
      {3-2*j-2*k,3-2*j-2*k,0,-3+2*j+2*k,-3+2*j+2*k,0},
      {1-2*j-k,-1-2*j,-2+k,-1+2*j+k,1+2*j,2-k}};
lu={2,1,-1,-2,-1,1};
a=a+(p+1)*If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,6]]],0]*
      lu[[mod[u,6]]]/72;
lu={1,-1};
a=a+If[p==3,ljk[[mod[j,3],mod[k,6]]],0]*lu[[mod[u,2]]]/18;
ljk={{0,-1,-1,0,1,1},{-2,-2,0,2,2,0},{-1,0,1,1,0,-1}};
lu={2,1,-1,-2,-1,1};
a=a-If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,6]]],0]*lu[[mod[u,6]]]/6;
lu={1,-1};
a=a-If[p==3,ljk[[mod[j,3],mod[k,6]]],0]*lu[[mod[u,2]]]/6;
```

(* contribution of $\varphi_8(3)$ and $\varphi_8(4)$ *)

(* contribution of $\varphi_{20}(3,r)$ and $\varphi_{20}(4,r)$ *)

```
ljk={{-1,1,0},{0,0,0},{0,1,-1}};
lu={2,-1,-1};
a=a+2/9*If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,3]]],0]*lu[[mod[u,3]]];
a=a+1/9*If[p==3,ljk[[mod[j,3],mod[k,3]]],0];
```

(* contribution of $\varphi_{21}(1,r)$ and $\varphi_{21}(2,r)$ *)

```
ljk={1,-1};
```

```
lu={1,-1};
a=a+(p+1)*(2*j+1)/128*ljk[[mod[j+k,2]]];
a=a+If[Mod[p,4]==1,(2*j+1)/64*ljk[[mod[j+k,2]]]*lu[[mod[u,2]]],0];
```

(* contribution of $\varphi_9(1)$ *)

```
ljk={{1,1,-1,-1},{-1,1,1,-1},{-1,-1,1,1},{1,-1,-1,1}};
a=a+If[p==2,ljk[[mod[j,4],mod[k,4]]]/16,0];
lu={4,-2,0,-2};
a=a+If[Mod[p,8]==1,ljk[[mod[j,4],mod[k,4]]]*lu[[mod[u,4]]]/16,0];
lu={2,-2};
a=a+If[Mod[p,8]==3,ljk[[mod[j,4],mod[k,4]]]*lu[[mod[u,2]]]/16,0];
lu={2,0,-2,0};
a=a+If[Mod[p,8]==5,ljk[[mod[j,4],mod[k,4]]]*lu[[mod[u,4]]]/16,0];
```

(* contribution of $\varphi_9(2)$ and $\varphi_9(3)$ *)

```
ljk={{0,-1,1},{-1,1,0},{1,0,-1}};
a=a+If[p==3,1,p+1]*(2*j+1)*ljk[[mod[j,3],mod[k,3]]]/108;
lu={2,-1,-1};
a=a+If[Mod[p,3]==1,(2*j+1)*ljk[[mod[j,3],mod[k,3]]],0]*
      lu[[mod[u,3]]]/108;
```

(* contribution of $\varphi_{10}(1)$ and $\varphi_{10}(2)$ *)

```
ljk={{1,-1},{-2,2},{1,-1}};
a=a+If[p==2,ljk[[mod[j,3],mod[k,2]]],0]/36;
lu={4,-1,1,-4,1,-1};
a=a+If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,2]]],0]*lu[[mod[u,6]]]/108;
lu={1,-1};
a=a+If[p==3,ljk[[mod[j,3],mod[k,2]]],0]*lu[[mod[u,2]]]/108;
```

(* contribution of $\varphi_{10}(3)$ *)

```
ljk={{2,1,-1,-2,-1,1},{-1,1,2,1,-1,-2},{-1,-2,-1,1,2,1}};
a=a+If[p==2,ljk[[mod[j,3],mod[k,6]]],0]/18;
lu={4,-1,1,-4,1,-1};
a=a+If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,6]]],0]*lu[[mod[u,6]]]/27;
lu={1,-1};
a=a+If[p==3,ljk[[mod[j,3],mod[k,6]]],0]*lu[[mod[u,2]]]/27;
```

(* contribution of $\varphi_{10}(i)$ $(i = 4, 5, 6, 7)$ *)

```
ljk={{0,1,-1},{-1,1,0},{-1,0,1},{0,-1,1},{1,-1,0},{1,0,-1}};
```

```
a=a+If[(p-2)*(p-3)==0,ljk[[mod[j,6],mod[k,3]]],0]/12;
lu={4,-3,1,0,1,-3};
a=a+If[Mod[p,12]==1,ljk[[mod[j,6],mod[k,3]]],0]*lu[[mod[u,6]]]/12;
lu={2,-2};
a=a+If[Mod[p,12]==5,ljk[[mod[j,6],mod[k,3]]],0]*lu[[mod[u,2]]]/12;
lu={2,-1,-1};
a=a+If[Mod[p,12]==7,ljk[[mod[j,6],mod[k,3]]],0]*lu[[mod[u,3]]]/12;
```

(* contribution of $\varphi_{10}(8)$ and $\varphi_{10}(9)$ *)

```
ljk={{1,0,0,0,-1,0,-1,0,0,0,1,0},{-1,0,0,0,1,0,1,0,0,0,-1,0}};
lu={4,0,2,0,-2,0,-4,0,-2,0,2,0};
a=a+If[Mod[p,12]==1,ljk[[mod[j,2],mod[k,12]]],0]*lu[[mod[u,12]]]/12;
ljk={{0,0,0,1,0,1,0,0,0,-1,0,-1},{0,-1,0,-1,0,0,0,1,0,1,0,0},
      {0,1,0,0,0,-1,0,-1,0,0,0,1}};
a=a-If[Mod[p,12]==1,ljk[[mod[j,3],mod[k,12]]],0]*lu[[mod[u,12]]]/12;
```

(* contribution of $\varphi_{11}(i)$ $(i = 1, 2, 3, 4)$ *)

```
ljk={{1,-1},{-2,2},{1,-1}};
a=a+If[p==2,ljk[[mod[j,3],mod[k,2]]],0]/36;
lu={4,-1,1,-4,1,-1};
a=a+If[Mod[p,3]==1,ljk[[mod[j,3],mod[k,2]]],0]*lu[[mod[u,6]]]/36;
lu={1,-1};
a=a+If[p==3,ljk[[mod[j,3],mod[k,2]]],0]*lu[[mod[u,2]]]/36;
```

(* contribution of φ_{12} *)

```
lj={1,-1,-1,1};
lk={1,-1};
a=a+If[p==2,lj[[mod[j,4]]]*lk[[mod[k,2]]],0]/16;
lu={4,-2,0,-2};
a=a+If[Mod[p,8]==1,lj[[mod[j,4]]]*lk[[mod[k,2]]]*lu[[mod[u,4]]],0]/16;
lu={2,-2};
a=a+If[Mod[p,8]==3,lj[[mod[j,4]]]*lk[[mod[k,2]]]*lu[[mod[u,2]]],0]/16;
lu={2,0,-2,0};
a=a+If[Mod[p,8]==5,lj[[mod[j,4]]]*lk[[mod[k,2]]]*lu[[mod[u,4]]],0]/16;
```

(* contribution of φ_{13} *)

```
ljk={{1,0,0,-1,0},{-1,1,0,0,0},{0,0,0,0,0},{0,0,0,1,-1},{0,-1,0,0,1}};
lu={4,-1,-1,-1,-1};
a=a+If[Mod[p,5]==1,ljk[[mod[j,5],mod[k,5]]],0]*lu[[mod[u,5]]]/5;
```

```
a=a+If[p==5,ljk[[mod[j,5],mod[k,5]]],0]/5;
(* contribution of φ14(i) (i = 1,2,3,4) *)

Return[a];

]
```

From the above theorem and the vanishing theorem (Theorem 6.1), we have the following

Corollary 4.12. *Let p, χ and u be as in the above theorem. If $j = 0$ and $k \geq 4$ or if $j > 0$ and $k \geq 5$, the dimension of $S_\mu(\Gamma_0(p), \chi)$ is equal to* `Siegel[p_,u_,j_,k_]`.

§5. Vector Bundle $\widetilde{\mathbf{V}}_{\boldsymbol{\mu}}$

Let $N \geq 3$ and μ a holomorphic representation of $GL(2, \mathbf{C})$. Let $X_2(N)$, $\overline{X}_2(N)$, $\widetilde{X}_2(N)$ and $\widetilde{V}_\mu$ be as in §2. $\widetilde{X}_2(N)$ has a natural morphism s: $\widetilde{X}_2(N) \to \overline{X}_2(N)$ which is the identity on $X_2(N)$. $\overline{X}_2(N)$ is set theoretically a disjoint union of $X_2(N)$, copies of $X_1(N)$'s which are called cusps of degree one and finite number of points which are called cusps of degree zero.

Let f be an element of $M_\mu(\Gamma_2(N)) \simeq H^0(X_2(N), \mathcal{O}(V_\mu))$. Then f has an extension $\tilde{f} \in H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu))$, since f has a Fourier expansion:

$$f(Z) = \sum_{S \geq O} a(S) \exp(2\pi i \mathrm{Tr}(SZ)/N)$$

at every cusp of degree one ([Gd]). Hence we have another isomorphism:

$$M_\mu(\Gamma_2(N)) \simeq H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu)).$$

We return to the case of general degree g in the following

Example 5.1. Let $Z \in \mathfrak{S}_g$ and put $Z = (Z_{ij})$, and let

$$\omega = \sum_{i \leq j} f_{ij}(Z) dZ_{ij}$$

be a holomorphic 1-form on $\mathfrak{S}_g$. We put $f_{ji}(Z) = f_{ij}(Z)$ and define a symmetric matrix valued holomorphic function f such that the (i,j)-coefficient of $f(Z)$ is equal to $f_{ij}(Z)$, if $i = j$ and $f_{ij}(Z)/2$, otherwise. Then ω is represented as $\mathrm{Tr}\, f(Z)(dZ_{ij})$. We put $W = M\langle Z \rangle$. Then since we have

$$(dW_{ij}) = {}^t(CZ+D)^{-1}(dZ_{ij})(CZ+D)^{-1},$$

ω is invariant under the action of $\Gamma_g(N)$, i. e.,

$$\sum_{i\le j} f_{ij}(M\langle Z\rangle)dW_{ij} = \sum_{i\le j} f_{ij}(Z)dZ_{ij},$$

for any $M \in \Gamma_g(N)$, if and only if f satisfies

$$f(M\langle Z\rangle) = (CZ+D)f(Z)^t(CZ+D)$$

for any M. Therefore in this case, f belongs to $M_{s_2}(\Gamma_g(N))$, where s_2 is the symmetric tensor representation of degree two of $GL(g, \mathbf{C})$. The 1-form $\widetilde{\omega}$ which corresponds to the extended section $\tilde{f} \in H^0(\widetilde{X}_g(N), \mathcal{O}(\widetilde{V}_{s_2}))$ of f may have logarithmic poles along the divisor at infinity D ([Mu]). Therefore we have the following isomorphism of vector bundles:

$$\mathcal{O}(\widetilde{V}_{s_2}) \simeq \Omega^1(\log D).$$

Remark 5.2. If $\mu(CZ+D) = \det(CZ+D)^k$, V_μ and $\widetilde{V}_\mu$ are line bundles which we also denote by L_2 and $\widetilde{L}_2$, respectively. L_2 can be extended to a holomorphic line bundle $\overline{L}_2$ on $\overline{X}_2(N)$, and $\widetilde{L}_2$ is the pullback of $\overline{L}_2$ by s. But in general V_μ cannot be extended onto $\overline{X}_2(N)$. The closure of a cusp of degree one of $\overline{X}_2(N)$ is biholomorphic to $\overline{X}_1(N)$ and the restriction of $\overline{L}_2$ to $\overline{X}_1(N)$ is isomorphic to $\overline{L}_1$.

Let

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}$$

be the coordinate system of $\mathfrak{S}_2$. Let C^0 be a cusp of degree one in $\overline{X}_2(N)$ and C its closure in $\overline{X}_2(N)$. Cusps of degree one in $\overline{X}_2(N)$ are equivalent to each other under the action of $\Gamma_2(1)/\Gamma_2(N)$. So we assume that C^0 is defined by $\operatorname{Im} Z_{22} = \infty$. Let D_1^0 be $s^{-1}(C^0)$ and D_1 the closure of D_1^0 in $\widetilde{X}_2(N)$. We denote the restriction of s to D_1 by π. $\widetilde{L}_2 \mid D_1$ is isomorphic to the pullback of $\overline{L}_1$ on $C \simeq \overline{X}_1(N)$ by π. D_1 has a structure of an elliptic surface over C and D_1^0 has the structure of the universal family of elliptic curves with level N ([Ig], [Nm] and [AMRY] Chapter I, §4).

We need to study the structure of $\widetilde{V}_\mu \mid D_1$. Let $p : \mathfrak{S}_1 \to X_1(N) \simeq C^0$ be the covering map. The universal covering space of D_1^0 is $\mathfrak{S}_1 \times \mathbf{C}$. Let $P(N)$ be the subgroup of $\Gamma_2(N)$ consisting of elements which map C into itself. The set

$$B(N) := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in N\mathbf{Z} \right\}$$

is a normal subgroup of $P(N)$. $P(N)/B(N)$ is isomorphic to the covering transformation group of the covering space $\widetilde{p} : \mathfrak{S}_1 \times \mathbf{C} \to D_1^0$. Let $M \in P(N)$. Then it is known that M is equivalent modulo $B(N)$ to an element such as

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & an-bm \\ m & 1 & n & 0 \\ c & 0 & d & cn-dm \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \quad \text{and} \quad m,\ n \in N\mathbf{Z}.$$

The coset $MB(N)$ acts on $\mathfrak{S}_1 \times \mathbf{C}$ as

$$(z_1, z_2) \quad \longmapsto \quad \left(\frac{az_1+b}{cz_1+d}, \frac{z_2+mz_1+n}{cz_1+d} \right),$$

for $(z_1, z_2) \in \mathfrak{S}_1 \times \mathbf{C}$. Let $\widetilde{\pi} : \mathfrak{S}_1 \times \mathbf{C} \to \mathfrak{S}_1$ be the projection to the first factor. Then $\pi \mid D_1^0 : D_1^0 \to X_1(N)$ is associated with $\widetilde{\pi}$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{S}_1 \times \mathbf{C} & \xrightarrow{\widetilde{p}} & D_1^0 \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi|D_1^0} \\ \mathfrak{S}_1 & \xrightarrow[p]{} & C^0. \end{array}$$

$\widetilde{V}_\mu \mid D_1^0$ is constructed directly as a quotient space of $\mathfrak{S}_1 \times \mathbf{C} \times \mathbf{C}^r$ by $P(N)/B(N)$ as follows. Let M be as above. Then the coset $MB(N)$ acts on $\mathfrak{S}_1 \times \mathbf{C} \times \mathbf{C}^r$ as

$$MB(N)((z_1, z_2), \xi) = \left(MB(N)(z_1, z_2), \mu \begin{pmatrix} cz_1+d & cn-dm \\ 0 & 1 \end{pmatrix} \xi \right), \tag{1}$$

for $(z_1, z_2) \in \mathfrak{S}_1 \times \mathbf{C}$ and $\xi \in \mathbf{C}^r$. $\widetilde{V}_\mu \mid D_1^0$ is biholomorphic to the quotient space of $\mathfrak{S}_1 \times \mathbf{C} \times \mathbf{C}^r$ by this action.

Let $(j+k, k)$ be the signature of μ where j and k are integers with $j \geq 0$. Then μ is of degree $j+1$ and equivalent to $s_j \otimes \det^k$. We have

$$s_j \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^j & \binom{j}{1} a^{j-1} b & \binom{j}{2} a^{j-2} b^2 & \cdots & b^j \\ 0 & a^{j-1} & \binom{j-1}{1} a^{j-2} b & \cdots & b^{j-1} \\ 0 & 0 & a^{j-2} & \cdots & b^{j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $j \geq 1$, this is written as

$$\begin{pmatrix} a^j & * \\ 0 & s_{j-1}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

Therefore from the explicit construction (1) of $\widetilde{V}_\mu \mid D_1^0$, we see that $\widetilde{V}_{s_{j-1}} \mid D_1^0$ is a quotient bundle of $\widetilde{V}_{s_j} \mid D_1^0$ and the kernel is the line bundle which corresponds to the automorphic factor $(cz_1 + d)^j$. Obviously this line bundle is isomorphic to the pullback of $L_1^{\otimes j}$ by $\pi \mid D_1^0$. Hence we have the following exact sequence of vector bundles:

$$0 \to \widetilde{L}_2^{\otimes j} \mid D_1^0 \to \widetilde{V}_{s_j} \mid D_1^0 \to \widetilde{V}_{s_{j-1}} \mid D_1^0 \to 0. \tag{2}$$

Let μ and σ be $s_j \otimes \det^k$ and $s_{j-1} \otimes \det^k$, respectively. Then multiplying by $\widetilde{L}_2^{\otimes k} \mid D_1^0$, we have another exact sequence:

$$0 \to \widetilde{L}_2^{\otimes (j+k)} \mid D_1^0 \to \widetilde{V}_\mu \mid D_1^0 \to \widetilde{V}_\sigma \mid D_1^0 \to 0. \tag{3}$$

We can easily see that this exact sequence is extended onto D_1. So we have the following exact sequence:

$$0 \to \widetilde{L}_2^{\otimes (j+k)} \mid D_1 \to \widetilde{V}_\mu \mid D_1 \to \widetilde{V}_\sigma \mid D_1 \to 0. \tag{4}$$

Therefore we derive

$$ch(\widetilde{V}_\mu \mid D_1) = ch(\widetilde{L}_2^{\otimes (j+k)} \mid D_1) + ch(\widetilde{V}_\sigma \mid D_1),$$

where ch means the Chern character and by induction we obtain the following

Theorem 5.3.

$$c_1(\widetilde{V}_\mu \mid D_1) = \sum_{i=0}^{j} (i+k) c_1(\widetilde{L}_2 \mid D_1) \tag{5}$$
$$= (1/2)(j+1)(j+2k) c_1(\widetilde{L}_2 \mid D_1).$$

$$c_2(\widetilde{V}_\mu \mid D_1) = 0. \tag{6}$$

Remark 5.4. Since $\det(\mu(CZ + D)) = \det(CZ + D)^{(j+1)(j+2k)/2}$, we have

$$c_1(\widetilde{V}_\mu) = (1/2)(j+1)(j+2k) c_1(\widetilde{L}_2).$$

So as to the first Chern class, (5) in the theorem holds without restricting to D_1. The author proved a similar exact sequence of vector bundles as (4) in the case of the Siegel upper half plane of degree three and $\mu = s_2$ by a different method (proof of [T1] Theorem (5.2)). We used that exact sequence to compute the dimension of $S_k(\Gamma_3(N))$ with $N \geq 3$. Such exact sequences are very important when we use the Riemann-Roch-Hirzebruch's formula (cf. proof of Theorem 5.10, below).

Let $\tilde{f} \in H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu))$ be the extension of $f \in H^0(X_2(N), \mathcal{O}(V_\mu))$, and let $\tilde{f} \mid D_1 \in H^0(D_1, \mathcal{O}(\widetilde{V}_\mu \mid D_1))$ be the restriction of $\tilde{f}$ to D_1. Now we have the following

Theorem 5.5. *In the above sequence (4), $\tilde{f} \mid D_1$ is mapped to* 0*, i. e.,*

$$\tilde{f} \mid D_1 \in H^0(D_1, \mathcal{O}(\widetilde{L}_2^{\otimes(j+k)} \mid D_1)).$$

Proof. Let $\tau \in \mathfrak{S}_1$. Then $\widetilde{p}(\{\tau\} \times \mathbf{C})$ is isomorphic to the elliptic curve $E_\tau := \mathbf{C}/N(\mathbf{Z}\tau + \mathbf{Z})$. $(N\mathbf{Z})^2$ acts on $\mathbf{C} \times \mathbf{C}^{(j+1)}$ as

$$(m, n)\,(w, \xi) = \left(w + m\tau + n, \mu\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}\xi\right),$$

for $w \in \mathbf{C}$, $\xi \in \mathbf{C}^{(j+1)}$ and $m,\, n \in N\mathbf{Z}$. $\widetilde{V}_\mu \mid E_\tau$ is biholomorphic to the quotient space of $\mathbf{C} \times \mathbf{C}^{(j+1)}$ by this action. If $j = 0$, $\mu(CZ + D) = \det(CZ + D)^k$ and

$$\mu\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} = 1.$$

So $\widetilde{V}_\mu \mid E_\tau$ is a trivial line bundle. This is the reason why L_2 is extended to $\overline{X}_2(N)$ and $\widetilde{L}_2 \mid D_1$ is isomorphic to $\pi^*(\overline{L}_1)$. Let $j \geq 1$. Then we have the following exact sequence of vector bundles from (4):

$$0 \to E_\tau \times \mathbf{C} \to \widetilde{V}_\mu \mid E_\tau \to \widetilde{V}_\sigma \mid E_\tau \to 0, \tag{7}$$

where $E_\tau \times \mathbf{C}$ means the trivial line bundle on E_τ. So to prove Theorem 5.5, it suffices to prove the following

Lemma 5.6. *The restriction $\tilde{f} \mid E_\tau$ of $\tilde{f}$ is mapped to* 0 *by the above sequence (7).*

Proof. Let $\phi = \widetilde{p} \mid \{\tau\} \times \mathbf{C}$. Then $\phi^*(\tilde{f} \mid E_\tau)$ is a section of $\phi^*(\widetilde{V}_\mu \mid E_\tau)$ which is identified with a holomorphic map a of $\mathbf{C}$ to $\mathbf{C}^{(j+1)}$ satisfying the following equalities:

$$\mu\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} a(w) = a(w + m\tau + n),$$

where

$$a(w) = \begin{pmatrix} a_j(w) \\ \vdots \\ a_1(w) \\ a_0(w) \end{pmatrix}.$$

We prove the assertion by induction on j. Let $j = 1$. Then we have the following relations:

$$a_1(w + m\tau + n) = a_1(w) - ma_0(w),$$
$$a_0(w + m\tau + n) = a_0(w).$$

From the second relation we derive that $a_0(w)$ is constant which we denote by α. Hence it follows that

$$a_1(w + m\tau + n) = a_1(w) - m\alpha.$$

Differentiating by w, we derive

$$a_1'(w + m\tau + n) = a_1'(w).$$

Therefore $a_1'(w)$ is a constant which we denote by β. So we have

$$a_1(w) = \beta w + \gamma,$$

where γ is a constant. Hence it follows that for any $w \in \mathbf{C}$ and $m,\, n \in N\mathbf{Z}$,

$$\beta(w + m\tau + n) + \gamma = \beta w + \gamma - m\alpha.$$

So we have $\alpha = \beta = 0$. Thus we proved that

$$a(w) = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}.$$

Now let $j \geq 2$. Then $a(w)$ is mapped to

$$\begin{pmatrix} a_{j-1}(w) \\ \vdots \\ a_1(w) \\ a_0(w) \end{pmatrix}.$$

By the assumption of the induction we have

$$a_0(w) \equiv a_1(w) \equiv \cdots \equiv a_{j-2}(w) \equiv 0.$$

Hence $a_j(w)$ and $a_{j-1}(w)$ satisfy the following relations:

$$a_j(w+m\tau+n) = a_j(w) - mja_{j-1}(w),$$
$$a_{j-1}(w+m\tau+n) = a_{j-1}(w).$$

So similarly as before we have

$$a_{j-1}(w) \equiv 0,$$

and $a_j(w)$ is a constant.

Let $j = 0$. Then we have the following isomorphism:

$$H^0(D_1, \mathcal{O}(\widetilde{L}_2^{\otimes k} \mid D_1)) \simeq H^0(\overline{X}_1(N), \mathcal{O}(\overline{L}_1^{\otimes k})). \tag{8}$$

For $f \in H^0(X_2(N), \mathcal{O}(L_2^{\otimes k}))$, we denote by $\Phi(f)$ the element of $H^0(\overline{X}_1(N), \mathcal{O}(\overline{L}_1^{\otimes k}))$ corresponding to $\tilde{f} \mid D_1$ by the above isomorphism (8). Let $j \geq 1$, and $f \in H^0(X_2(N), \mathcal{O}(V_\mu))$. Then by Theorem 5.5, $\tilde{f} \mid D_1$ belongs to $H^0(D_1, \mathcal{O}(\widetilde{L}_2^{\otimes(j+k)} \mid D_1))$. We denote by $\Phi(f)$ the element of $H^0(\overline{X}_1(N), \mathcal{O}(\overline{L}_1^{\otimes(j+k)}))$ which corresponds to $\tilde{f} \mid D_1$ by the above isomorphism (8).

Definition 5.7. The linear map:

$$\Phi : M_\mu(\Gamma_2(N)) \simeq H^0(X_2(N), \mathcal{O}(V_\mu)) \to H^0(\overline{X}_1(N), \mathcal{O}(\overline{L}_1^{\otimes(j+k)})) \simeq M_{j+k}(\Gamma_1(N))$$

is called Φ-*operator*. Φ-operator is defined for each cusp of degree one and $f \in M_\mu(\Gamma_2(N))$ is called *a cusp form*, if f belongs to the kernel of Φ-operator for each cusp of degree one. We denote by $S_\mu(\Gamma_2(N))$ the subspace of $M_\mu(\Gamma_2(N))$ consisting of cusp forms. If $\Gamma \supset \Gamma_2(N)$, $S_\mu(\Gamma)$ is defined to be $M_\mu(\Gamma) \cap S_\mu(\Gamma_2(N))$.

Remark 5.8. In [Gd] Φ-operator is defined by a different method. We return to the general case of degree g and recall the definition of Φ-operator there. Let $F_\mu^{(g)}$ be the representation space of a holomorphic representation μ of $GL(g, \mathbf{C})$. Let $F_\mu^{(g')}$ be the subspace of $F_\mu^{(g)}$ consisting of elements which are fixed by

$$\mu \begin{pmatrix} 1_{g'} & M_1 \\ 0 & M_2 \end{pmatrix},$$

for any M_1 and M_2 with $\det M_2 = 1$. Then we have

$$F_\mu^{(g)} \supseteq F_\mu^{(g-1)} \supseteq \cdots \supseteq F_\mu^{(1)} \supseteq F_\mu^{(0)}.$$

$F_\mu^{(g')}$ is stable under the action of

$$\mu^{(g')}(M) := \mu \begin{pmatrix} M & 0 \\ 0 & 1_{g-g'} \end{pmatrix},$$

for any $M \in GL(g', \mathbf{C})$. $\Phi^{(g-1)}$ is defined by

$$\Phi^{(g-1)} f(Z_1) = \lim_{\operatorname{Im} Z_{gg} \to \infty} f \begin{pmatrix} Z_1 & u \\ {}^t u & Z_{gg} \end{pmatrix},$$

for $Z_1 \in \mathfrak{S}_{g-1}$. $\Phi^{(g')}$ is defined inductively for general g'. If $Z \in \mathfrak{S}_{g'}$, $\Phi^{(g')} f(Z)$ belongs to $F_\mu^{(g')}$ and $\Phi^{(g')} f$ is an automorphic form of type $\mu^{(g')}$ with respect to $\Gamma_{g'}(N)$.

In the case of degree two, our Φ-operator coincides with $\Phi^{(1)}$. If $j = 0$, $F_\mu^{(2)}, F_\mu^{(1)}$ and $F_\mu^{(0)}$ coincide with each other, and if $j \geq 1$, $F_\mu^{(1)}$ is one-dimensional and $F_\mu^{(0)}$ is zero-dimensional.

The following proposition is obvious from the observation above.

Proposition 5.9. *$f \in H^0(X_2(N), \mathcal{O}(V_\mu))$ is a cusp form if and only if $\tilde{f}$ vanishes along D. Therefore we have*

$$S_\mu(\Gamma_2(N)) \simeq H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu - D)),$$

where $\mathcal{O}(\widetilde{V}_\mu - D)$ is the subsheaf of $\mathcal{O}(\widetilde{V}_\mu)$ consisting of germs of sections of $\widetilde{V}_\mu$ which vanish along D.

Now we can calculate $\dim S_\mu(\Gamma_2(N))$ $(N \geq 3)$.

Theorem 5.10. *Let $N \geq 3$. If $j = 0$ and $k \geq 4$ or if $j \geq 1$ and $k \geq 5$, then the dimension of $S_\mu(\Gamma_2(N))$ is equal to*

$$2^{-8}3^{-3}5^{-1}((j+1)(k-2)(j+k-1)(j+2k-3)N^{10} - 60(j+1)(j+2k-3)N^8 + 360(j+1)N^7) \prod_{p|N,\ p:prime} (1-p^{-2})(1-p^{-4}).$$

Proof. Let $[D]$ be the line bundle on $\widetilde{X}_2(N)$ which is associated with the divisor D. Then by the vanishing theorem (Theorem 6.1, below), we have

$$\begin{aligned} \dim S_\mu(\Gamma_2(N)) &= \dim H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu - D)) \\ &= \dim H^0(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu \otimes [D]^{\otimes(-1)})) \\ &= \chi(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu \otimes [D]^{\otimes(-1)})), \end{aligned}$$

where χ means the Euler-Poincaré characteristic. Let c_i be the i-th Chern class of $\widetilde{X}_2(N)$ and let $\overline{c}_i$ be the i-th logarithmic Chern class of $X_2(N)$ with respect to $\widetilde{X}_2(N)$ ([T1]) and $S^2(D)$ the second fundamental symmetric polynomial of (the cohomology classes of) the irreducible divisors in D. Then from [T1] Proposition (1.2), we have

$$
\begin{aligned}
c_1 &= \overline{c}_1 + D \\
c_2 &= \overline{c}_2 + \overline{c}_1 D + S^2(D),
\end{aligned}
$$

where we denote the divisor D and its cohomology class by the same notation. By the formula of Riemann-Roch-Hirzebruch, we have

$$
\begin{aligned}
&\chi(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu \otimes [D]^{\otimes(-1)})) \\
=&\frac{1}{24}(4c_1(\widetilde{V}_\mu)^3 + 12c_3(\widetilde{V}_\mu) - 12c_1(\widetilde{V}_\mu)c_2(\widetilde{V}_\mu) + 6c_1(\widetilde{V}_\mu)^2\overline{c}_1 - 12c_2(\widetilde{V}_\mu)\overline{c}_1 \\
&\qquad + 2c_1(\widetilde{V}_\mu)\overline{c}_1^2 + 2c_1(\widetilde{V}_\mu)\overline{c}_2 + (j+1)\overline{c}_2\overline{c}_1) \\
&+ \frac{1}{24}(12c_2(\widetilde{V}_\mu) - 6c_1(\widetilde{V}_\mu)\overline{c}_1 - 6\overline{c}_1^2 - (j+1)(\overline{c}_1^2 + \overline{c}_2))D \\
&+ \frac{1}{24}((j+1)\overline{c}_1 D^2 + 2c_1(\widetilde{V}_\mu)D^2) + \frac{1}{24}(2c_1(\widetilde{V}_\mu)S^2(D) + (j+1)\overline{c}_1 S^2(D)) \\
&- \frac{1}{24}(j+1)D\, S^2(D).
\end{aligned}
$$

Since

$$\bar{c}_j = (-1)^j c_j(\widetilde{V}_{s_2})$$

by Example 5.1, the terms in the first and the second lines are proportional to the invariant volume of $\Gamma_2(N)\backslash\mathfrak{S}_2$ ([Mu]). (In [Mu] Theorem 3.2, it is stated that a polynomial of $c_i(\widetilde{V}_\mu)$'s for a single representation μ is proportional. But moreover a polynomial of $c_i(\widetilde{V}_\mu)$'s for various representations is also proportional. Proof is the same.) Therefore these terms are calculated by [Is] Theorem 4. Note that "the canonical factor of automorphy" in [Is] is ${}^t(CZ+D)^{-1}$ by our notation and [Is] Theorem 4 ii) is misprinted. "$\overset{\circ}{\sigma}{}' + \overset{\circ}{\delta}_K$" should be "$\overset{\circ}{\sigma}{}' - \overset{\circ}{\delta}_K$".

The terms in the third line and the second term in the fourth line vanish by Theorem 5.3. Next from (5) in Theorem 5.3, we have

$$c_1(\widetilde{V}_\mu)D^2 = (1/2)(j+1)(j+2k)c_1(\widetilde{L}_2)D^2.$$

and it holds that $\overline{c}_1 = -3c_1(\widetilde{L}_2)$. Hence the first term in the fourth line is equal to

$$(1/24)(j+1)(j+2k-3)c_1(\widetilde{L}_2)D^2.$$

This and the term in the fifth line are similarly calculated as in [T1].

§6. Proof of the Vanishing Theorem

In this section we prove the vanishing theorem:

Theorem 6.1. *Let μ be an irreducible holomorphic representation of $GL(2,\mathbf{C})$ and let $(j+k,k)$ be its signature. If $j=0$ and $k\geq 4$ or if $j\geq 1$ and $k\geq 5$ and if $p>0$, then*

$$H^p(\widetilde{X}_2(N),\mathcal{O}(\widetilde{V}_\mu - D))\simeq 0.$$

We reduce the proof of this theorem to the case of the cohomology group of a certain line bundle of a $\mathbf{P}^1$-bundle over $\widetilde{X}_2(N)$ and apply the vanishing theorem of Kawamata-Viehweg ([Ka] and [V]). But this proof is a rather makeshift one. The vector bundle $\mathcal{V}_\mu$ on $\mathfrak{S}_2$ has a $Sp\,(2,\mathbf{R})$-invariant hermitian metric which induces a metric on $\widetilde{V}_\mu\otimes[D]^{\otimes(-1)}$. This metric "degenerates" along the divisor at infinity D. If one develop a theory of harmonic integrals with respect such a degenerating metric (cf. [Z] for one dimensional case), then our vanishing theorem can be proved by applying the vanishing theorem of Nakano directly to the vector bundle $\widetilde{V}_\mu\otimes[D]^{\otimes(-1)}$. Before the proof of Theorem 6.1, we present a proof (which uses the vanishing theorem of Nakano ([Nk])) of the vanishing theorem in case $\Gamma\backslash\mathfrak{S}_g$ is compact, because we have to use the positivity of the $Sp\,(2,\mathbf{R})$-invariant metric in the proof of Theorem 6.1.

Remark 6.2. In the case of compact quotients, the vanishing theorem holds when $k=4$ (Theorem 6.6). So it is expected that the vanishing theorem also holds when $k=4$ in the case of non-compact quotients.

Now we prove the vanishing theorem for the case of compact quotients by using Nakano's vanishing theorem. First we recall the positivity of holomorphic vector bundles in the sense of Nakano. Let X be a complex manifold of dimension n, V a holomorphic vector bundle of rank r on X and h a hermitian metric on V. Put

$$\theta=h^{-1}\partial h \quad\text{and}\quad \Theta=\bar{\partial}\theta.$$

θ is the connection form and Θ is the curvature form. Θ is a $r\times r$ matrix whose coefficients are $(1,1)$-forms. Let $(z^1,z^2,\dots,z^n)$ be the local coordinate in X and let

$$\sum_{1\leq i,j\leq n} H(\Theta)_{\sigma i,\rho j}dz^i d\bar{z}^j$$

be the (ρ, σ)-coefficient of $h\Theta$. Since it holds that

$$H(\Theta)_{\sigma i, \rho j} = \overline{H(\Theta)}_{\rho j, \sigma i},$$

we can define a hermitian form:

$$\Theta(\xi, \xi) = \sum_{\substack{1 \leq \sigma, \rho \leq r \\ 1 \leq i, j \leq n}} H(\Theta)_{\sigma i, \rho j} \xi^{\sigma i} \bar{\xi}^{\rho j}$$

for a vector $\xi = (\xi^{\sigma i})_{1 \leq \sigma \leq r, 1 \leq i \leq n}$.

Definition 6.3. ([Nk]). If the hermitian form $\Theta(\xi, \xi)$ is positive (resp. non-negative or negative) for every non-zero ξ, V is said to be positive (resp. semi-positive or negative[5]) and written as

$$V > 0 \quad \text{(resp. } V \geq 0 \ \text{ or } \ V < 0).$$

In the case of the line bundle this notion of positivity coincides with the positivity in the sense of Kodaira ([Ko]).

Now we return to the case of the Siegel upper half plane. Let μ be the representation of $GL(g, \mathbf{C})$ on $\mathbf{C}^g$ with the standard action. In this case we denote $\mathcal{V}_\mu$, V_μ and $\widetilde{V}_\mu$ by $\mathcal{V}$, V and $\widetilde{V}$, respectively. Let $Z \in \mathfrak{S}_g$. For $u,\ v \in \mathcal{V}_Z$, we put

$$\mathcal{H} = \operatorname{Im} Z \quad \text{and} \quad \langle u, v \rangle = {}^t\bar{u}\mathcal{H}v.$$

Then since it holds that

$$\operatorname{Im} M \langle Z \rangle = {}^t\overline{(CZ + D)}^{-1} (\operatorname{Im} Z)(CZ + D)^{-1},$$

for any $Z \in \mathfrak{S}_g$ and $M \in Sp\,(g, \mathbf{R})$, we have

$$\langle Mu, Mv \rangle = \langle u, v \rangle\,,$$

i. e., this hermitian metric $\mathcal{H}$ is $Sp\,(g, \mathbf{R})$-invariant. Therefore for any torsion free discrete subgroup Γ of $Sp\,(g, \mathbf{R})$, this metric $\mathcal{H}$ induces a hermitian metric h on $V := \Gamma \backslash \mathcal{V}$. In case $\mu(CZ + D) = \det(CZ + D)$ we also denote $\mathcal{V}_\mu$ by $\mathcal{L}_g$. Similarly as before, we define $\langle u, v \rangle = \bar{u}(\det \mathcal{H})v$ for $Z \in \mathfrak{S}_g$ and $u, v \in (\mathcal{L}_g)_Z$, and this metric is also $Sp\,(g, \mathbf{R})$-invariant. Therefore this metric induces a metric on $L_g := \Gamma \backslash \mathcal{L}_g$.

[5] The definition of the negativity was false. V is said to be negative if V^* is positive.

Proposition 6.4. *With respect to the above metric, we have*

$$\mathcal{V} \geq 0, \tag{9}$$

$$\mathcal{L}_g > 0. \tag{10}$$

Proof. Since the metric $\mathcal{H}$ is $Sp\,(g, \mathbf{R})$-invariant, it suffices to prove the positivity at a single point $Z \in \mathfrak{S}_g$. We put $Z = \sqrt{-1}1_g$. Let $Y = \mathrm{Im}\, Z$. Then

$$\begin{aligned}(\Theta_{ij}) &= \bar{\partial}(Y^{-1}\partial Y) \\ &= Y^{-1}\bar{\partial}\partial Y - Y^{-1}\bar{\partial}Y Y^{-1}\partial Y \\ &= -(1/4)(\sum_k d\overline{Z}_{ik} \wedge dZ_{kj}).\end{aligned}$$

We introduce variables $\xi^{\sigma(i,j)}$ for $1 \leq \sigma, i, j \leq g$ and $i \leq j$, and put $\xi^{\sigma(j,i)} = \xi^{\sigma(i,j)}$. Then for $\xi = (\xi^{\sigma(i,j)})$ it holds that

$$\begin{aligned}\Theta(\xi, \xi) &= (1/4) \sum_{1 \leq i,j,k \leq g} \xi^{j(j,k)} \bar{\xi}^{i(i,k)} \\ &= (1/4) \sum_{1 \leq k \leq g} \sum_{1 \leq i,j \leq g} \xi^{j(j,k)} \bar{\xi}^{i(i,k)} \\ &= (1/4) \sum_{1 \leq k \leq g} \Big| \sum_{1 \leq i \leq g} \xi^{i(i,k)} \Big|^2 \\ &\geq 0.\end{aligned}$$

So (9) is proved. (10) is well known, since $\sqrt{-1}\bar{\partial}\partial \log(\det \mathcal{H})$ defines the Bergmann metric on $\mathfrak{S}_g$.

The following lemma is proved similarly as in [Gr2] p.209, although the positivity in [Gr2] is different from our positivity. (Actually the positivity in [Gr2] is weaker than ours.)

Lemma 6.5. *Let V_1 and V_2 be holomorphic vector bundles on a complex manifold X. If $V_1 \geq 0$ and $V_2 \geq 0$ with respect to some hermitian metrics h_1 and h_2, $V_1 \otimes V_2 \geq 0$ with respect to the metric $h_1 \otimes h_2$. Moreover if $V_2 > 0$, then $V_1 \otimes V_2 > 0$.*

Let μ be an irreducible holomorphic representation of $GL(g, \mathbf{C})$ and $(f_1, f_2, \ldots, f_g)$ with $f_1 \geq f_2 \geq \cdots \geq f_g$ its signature. Let Γ be a discrete subgroup of $Sp\,(g, \mathbf{R})$ without torsion elements such that $\Gamma \backslash \mathfrak{S}_g$ is compact. In this case, we have the following

Theorem 6.6. *If* $f_g \geq g+2$ *and if* $p > 0$*, then*

$$H^p(\Gamma\backslash\mathfrak{S}_g, \mathcal{O}(V_\mu)) \simeq 0.$$

Proof. Let K be the canonical line bundle on $\Gamma\backslash\mathfrak{S}_g$. Then it is known that

$$K \simeq L_g^{\otimes(g+1)}.$$

Let σ be the irreducible representation of $GL(g, \mathbf{C})$ whose signature is

$$(f_1 - g - 2, f_2 - g - 2, \ldots, f_g - g - 2).$$

Then $\mu = \sigma \otimes \det^{(g+2)}$. Since $f_g - g - 2 \geq 0, V^{\otimes f}$ contains a vector bundle isomorphic to V_σ as its direct summand, where f is equal to $\sum_{i=1}^{g}(f_i - g - 2)$. Therefore it suffices to prove that

$$H^p(\Gamma\backslash\mathfrak{S}_g, \mathcal{O}(V^{\otimes f} \otimes L_g^{\otimes(g+2)})) \simeq 0.$$

Since $V^{\otimes f} \otimes L_g^{\otimes(g+2)}$ is isomorphic to $V^{\otimes f} \otimes L_g \otimes K$, this is proved by Proposition 6.4, Lemma 6.5 and the following

Theorem 6.7. ([Nk]) *Let* X *be a compact complex manifold,* V *a holomorphic vector bundle on* X *and* K *the canonical line bundle on* X*. If* $V > 0$ *and if* $p > 0$*, then*

$$H^p(X, \mathcal{O}(V \otimes K)) \simeq 0.$$

Remark 6.8. Theorem 6.6 was proved for general bounded symmetric domains in [Is] and [MM]. The dimension of the spaces of vector valued automorphic forms in the case of compact quotients was calculated in [Is] for general bounded symmetric domain (and also in [L] by the Selberg's trace formula).

Now we return to the proof of Theorem 6.1. Let W be a holomorphic vector bundle of rank r on a compact complex manifold X and let W^* be its dual vector bundle. We identify X with the zero section of W^*. Then $\mathbf{P}(W) := (W^* - X)/\mathbf{C}^*$ is a $\mathbf{P}^{r-1}$ bundle on X and $W^* - X$ is a $\mathbf{C}^*$ bundle on $\mathbf{P}(W)$. We denote by $H(W)^*$ the tautological line bundle on $\mathbf{P}(W)$ which is the line bundle associated with this $\mathbf{C}^*$ bundle and we denote by $H(W)$ its dual line bundle.

Definition 6.9. Let L be a holomorphic line bundle on a projective manifold X. L is said to be *numerically semi-positive* if for any non-singular compact curve B and any holomorphic map $f : B \to X$, the degree of $f^*(L)$ is non-negative. Let W be a holomorphic vector bundle on X. W is said to be *numerically semi-positive* if $H(W)$ is numerically semi-positive.

The following isomorphism is well known:

$$H^p(X, \mathcal{O}(S^j(W)) \otimes \mathcal{F}) \simeq H^p(\mathbf{P}(W), \mathcal{O}(H(W)^{\otimes j}) \otimes \varpi^*\mathcal{F}), \tag{11}$$

where $\mathcal{F}$ is a coherent sheaf on X and $\varpi : \mathbf{P}(W) \to X$ is the projection. Let K_X and $K_{\mathbf{P}(W)}$ be the canonical line bundles on X and $\mathbf{P}(W)$, respectively. Then we have the following isomorphism ([Gr2] p.202 or [KO]):

$$K_{\mathbf{P}(W)} \simeq H(W)^{\otimes(-r)} \otimes \varpi^*(K_X \otimes (\det(W))). \tag{12}$$

In the following we denote $\widetilde{V}_\mu$ for the standard representation μ of $GL(2, \mathbf{C})$ on $\mathbf{C}^2$ by $\widetilde{V}$. Let μ be $s_j \otimes \det^k$. Then $\widetilde{V}_\mu$ is isomorphic to $S^j(\widetilde{V}) \otimes \widetilde{L}_2^{\otimes k}$. The case of $j = 0$ is easily proved by the ampleness of $\overline{L}_2$ ([Ba]) and Kodaira vanishing theorem ([Ko]). So we assume that $j \geq 1$. Then $\mathcal{O}(\widetilde{V}_\mu - D)$ is isomorphic to $\mathcal{O}(\widetilde{V}_\mu \otimes [D]^{\otimes(-1)})$. So this is isomorphic to $\mathcal{O}(S^j(\widetilde{V}) \otimes \widetilde{L}_2^{\otimes k} \otimes [D]^{\otimes(-1)})$. Let K be the canonical line bundle on $\widetilde{X}_2(N)$. Then K is isomorphic to $\widetilde{L}_2^{\otimes 3} \otimes [D]^{\otimes(-1)}$. So the above sheaf is isomorphic to $\mathcal{O}(S^j(\widetilde{V}) \otimes \widetilde{L}_2^{\otimes(k-3)} \otimes K)$. By the isomorphism (9), we have the following isomorphism:

$$H^p(\widetilde{X}_2(N), \mathcal{O}(S^j(\widetilde{V}) \otimes \widetilde{L}_2^{\otimes(k-3)} \otimes K)) \simeq H^p(\mathbf{P}(\widetilde{V}), \mathcal{O}(H(\widetilde{V})^{\otimes j} \otimes \varpi^*(\widetilde{L}_2^{\otimes(k-3)} \otimes K))).$$

Since $\widetilde{L}_2 = \det(\widetilde{V})$, this is isomorphic to

$$H^p(\mathbf{P}(\widetilde{V}), \mathcal{O}(H(\widetilde{V})^{\otimes(j+2)} \otimes \varpi^*(\widetilde{L}_2^{\otimes(k-4)}) \otimes K_{\mathbf{P}(\widetilde{V})})), \tag{13}$$

from (12). We prove that this cohomology group vanishes by the following theorem of Kawamata-Viehweg:

Theorem 6.10. ([Ka], [V]). *Let X be a projective manifold of dimension n, L a holomorphic line bundle on X, $c_1(L)$ the first Chern class of L and K the canonical line bundle on X. If L is numerically semi-positive and $c_1(L)^n[X]$ is positive, then we have*

$$H^p(X, \mathcal{O}(L \otimes K)) \simeq 0,$$

for $p > 0$.

So we have to prove the numerical semi-positivity of $L := H(\widetilde{V})^{\otimes(j+2)} \otimes \varpi^*(\widetilde{L}_2^{\otimes(k-4)})$ and the positivity of $c_1(L)^4[\mathbf{P}(\widetilde{V})]$ for $k \geq 5$. Since $\overline{L}_2$ is an ample line bundle on $\overline{X}_2(N)$ and $\varpi^*(\widetilde{L}_2)$ is the pullback of $\overline{L}_2$ by $s \circ \varpi : \mathbf{P}(\widetilde{V}) \to \widetilde{X}_2(N) \to \overline{X}_2(N)$, this is numerically semi-positive. Therefore it suffices to prove the numerical semi-positivity of $H(\widetilde{V})$ and the positivity of $c_1(L)^4[\mathbf{P}(\widetilde{V})]$.

First we calculate $c_1(L)^4[\mathbf{P}(\widetilde{V})]$. Put $a = j + 2$ and $b = k - 4$, and let e_1 and e_2 be the first and the second Chern classes of $\widetilde{V}$, respectively. Then by [Gr1] (A.9), we have

$$c_1(H(\widetilde{V}))^2 - \varpi^*(e_1)c_1(H(\widetilde{V})) + \varpi^*(e_2) = 0.$$

Since $e_1 = c_1(\widetilde{V}) = c_1(\widetilde{L}_2)$, we have

$$\begin{aligned} c_1(L)^4[\mathbf{P}(\widetilde{V})] &= (ac_1(H(\widetilde{V})) + b\varpi^*(c_1(\widetilde{L}_2)))^4[\mathbf{P}(\widetilde{V})] \\ &= \varpi^*((a^4 + 4a^3b + 6a^2b^2 + 4ab^3)e_1^3 - 2(a^4 + 2a^3b)e_1e_2)c_1(H(\widetilde{V}))[\mathbf{P}(\widetilde{V})] \\ &= ((a^4 + 4a^3b + 6a^2b^2 + 4ab^3)e_1^3 - 2(a^4 + 2a^3b)e_1e_2)[\widetilde{X}_2(N)] \end{aligned}$$

Let h be the metric on V which is induced by the $Sp\,(2, \mathbf{R})$-invariant metric $\mathcal{H}$, and let Θ be the curvature form of h. Put

$$\det\left(\mathbf{1}_2 + \frac{1}{2\pi i}\Theta\right) = \omega_0 + \omega_1 + \omega_2,$$

where ω_i is (i,i)-form. Since h degenerates along D, h does not define the metric on $\widetilde{V}$. But h is *good on* $\widetilde{X}_2(N)$ in the sense of [Mu] p.242. Namely h is dominated by the Poincaré metric on $\widetilde{X}_2(N)$ ([Mu] p.240). Hence ω_i is locally integrable on $\widetilde{X}_2(N)$ and this defines a current $[\omega_i]$ on $\widetilde{X}_2(N)$, and this current represents the cohomology class e_i ([Mu]). Since h is induced by the $Sp\,(g, \mathbf{R})$-invariant metric $\mathcal{H}$, ω_1 and ω_2 are induced by $Sp\,(g, \mathbf{R})$-invariant differential forms Ω_1 and Ω_2 on $\mathfrak{S}_2$, respectively. From [BH] §16.4, we have $\Omega_1 \wedge \Omega_1 = 2\,\Omega_2$. (We can prove this directly. At $Z = \sqrt{-1}1_2$, we have

$$\begin{aligned} \Omega_1 &= \frac{1}{8\pi i}\left(dZ_{11} \wedge d\overline{Z}_{11} + 2\,dZ_{12} \wedge d\overline{Z}_{12} + dZ_{22} \wedge d\overline{Z}_{22}\right), \\ \Omega_2 &= \left(\frac{1}{8\pi i}\right)^2 \left(dZ_{11} \wedge d\overline{Z}_{11} \wedge dZ_{22} \wedge d\overline{Z}_{22} + 2\,dZ_{11} \wedge d\overline{Z}_{11} \wedge dZ_{12} \wedge d\overline{Z}_{12}\right. \\ &\qquad \left. +2\,dZ_{12} \wedge d\overline{Z}_{12} \wedge dZ_{22} \wedge d\overline{Z}_{22}\right), \end{aligned}$$

(cf. proof of Proposition 6.4).) Hence we have $\omega_1 \wedge \omega_1 = 2\,\omega_2$ and $\omega_1 \wedge \omega_1 \wedge \omega_1 = 2\,\omega_1 \wedge \omega_2$. Since $\Omega_1 \wedge \Omega_1 \wedge \Omega_1$ is a multiple of the invariant volume form by a constant, we can calculate

$e_1^3[\widetilde{X}_2(N)]$ from the invariant volume of the fundamental domain of $\Gamma_2(N)$ in $\mathfrak{S}_2$. This is essentially the Hirzebruch proportionality ([Mu]). Therefore from [T1] Corollary (1.6), we have

$$e_1^3[\widetilde{X}_2(N)] = 2e_1e_2[\widetilde{X}_2(N)] = 2^{-6}3^{-2}5^{-1}N^{10}\prod(1-p^{-2})(1-p^{-4}),$$

where $\prod$ means $\prod_{p|N,\, p:prime}$. So $c_1(L)^4[\mathbf{P}(\widetilde{V})]$ is equal to

$$2^{-5}3^{-2}5^{-1}ab(a+b)(a+2b)N^{10}\prod(1-p^{-2})(1-p^{-4}).$$

Therefore $c_1(L)^4[\mathbf{P}(\widetilde{V})]$ is positive if $k \geq 5$.

Now we prove the numerical semi-positivity of $H(\widetilde{V})$. Let $\widetilde{V}^*$ be the dual vector bundle of $\widetilde{V}$. If $Z \in X_2(N)$, then $\widetilde{V}_Z^*$ has a hermitian form $h^* := {}^t h^{-1}$. Let u be a non-zero element of $\widetilde{V}_Z^*$. We define a positive function $\widehat{h}$ on $\widetilde{V}^* \mid X_2(N)$ minus its zero section by $\widehat{h}(Z,u) = {}^t\bar{u}h^*u$. For $\lambda \in \mathbf{C}^*$, it holds that $\widehat{h}(Z,\lambda u) = |\lambda|^2\widehat{h}(Z,u)$. Since $H(\widetilde{V})^*$ minus its zero section is naturally biholomorphic to $\widetilde{V}^*$ minus its zero section, $\widehat{h}$ defines a metric on $H(\widetilde{V})^* \mid \varpi^{-1}(X_2(N))$. We also denote this metric by $\widehat{h}$. Since $\widetilde{V} \mid X_2(N) \geq 0$ with respect to the metric h by Proposition 6.4, we have $\widetilde{V}^* \mid X_2(N) \leq 0$ with respect to h^*. Therefore $H(\widetilde{V})^* \mid \varpi^{-1}(X_2(N)) \leq 0$ with respect to $\widehat{h}$. These facts are similarly proved as in [Gr2] or [KO].

We regard $\varpi^{-1}(D)$ as the boundary (or points at infinity) of $\varpi^{-1}(X_2(N))$. Since h^* is induced from the $Sp(2,\mathbf{R})$-invariant metric $\mathcal{H}^* := {}^t\mathcal{H}^{-1}$, we can easily see that the metric $\widehat{h}$ is good on $\mathbf{P}(\widetilde{V})$. Namely $\widehat{h}$ is dominated by the Poincaré metric on $\mathbf{P}(\widetilde{V})$. Let

$$\omega = \frac{1}{2\pi i}\partial\bar{\partial}\log\widehat{h}(Z,u).$$

Then ω is locally integrable on $\mathbf{P}(\widetilde{V})$ and the current $[\omega]$ represents the first Chern class of $H(\widetilde{V})^*$.

Let B be a compact smooth curve and $f : B \to \mathbf{P}(\widetilde{V})$ a holomorphic map. We prove that the degree of $f^*(H(\widetilde{V}))$ is non-negative. First we assume that $f(B) \not\subset D$. We put $B^0 = f^{-1}(\varpi^{-1}(X_2(N)))$. We regard $B - B^0$ as the boundary of B^0. We define a metric h_B on $f^*(H(V)^*) \mid B^0$ as the pullback of $\widehat{h}$ by f. Let $\omega^{(p)}$ be the Poincaré metric on $\mathbf{P}(\widetilde{V})$. Then $f^*(\omega^{(p)})$ is dominated by the Poincaré metric on B^0 and so is h_B. Therefore h_B is good on B. Hence

$$\omega_B := \frac{1}{2\pi i}\partial\bar{\partial}\log h_B \ (= f^*(\omega))$$

is locally integrable on B and the current $[\omega_B]$ represents the first Chern class of $f^*(H(\widetilde{V})^*)$. So we have

$$\deg f^*(H(\widetilde{V})^*) = \int_{B^0} \omega_B.$$

Since $H(\widetilde{V})^* \mid \varpi^{-1}(X_2(N)) \leq 0$ with respect to $\widehat{h}$, we have $f^*(H(\widetilde{V})^*) \mid B^0 \leq 0$ with respect to h_B. Hence the above integral is non-positive. Therefore the degree of $f^*(H(\widetilde{V}))$ is non-negative.

Next we assume that $f(B) \subset D$. Let D_1 be the irreducible component of D which contains $f(B)$. Then as we saw in §5, we have the following exact sequence of vector bundles:

$$0 \to \widetilde{L}_2 \mid D_1 \to \widetilde{V} \mid D_1 \to D_1 \times \mathbf{C} \to 0,$$

where $D_1 \times \mathbf{C}$ means the trivial line bundle on D_1. We need to prove the numerical semi-positivity of $\widetilde{V} \mid D_1$. This is proved by the following

Theorem 6.11. *Let X be a projective manifold and let*

$$0 \to V_1 \overset{\phi}{\to} V_2 \overset{\psi}{\to} V_3 \to 0.$$

be an exact sequence of holomorphic vector bundles on X. If V_1 and V_3 are numerically semi-positive, then V_2 is also numerically semi-positive.

Proof. (due to T. Fujita). A holomorphic vector bundle V on X is numerically semi-positive if and only if for any compact smooth curve B, any holomorphic map $f : B \to X$ and any quotient line bundle L of $f^*(V)$, the degree of L is non-negative. This is similarly proved as [F] Proposition (2.8). Let $f : B \to X$ be as above, L a line bundle on B and

$$\theta : f^*(V_2) \to L \to 0$$

an exact sequence. Let $\theta \circ f^*(\phi)$ be the composition of θ and $f^*(\phi)$ where $f^*(\phi)$ is the pullback of ϕ by f and let r be the rank of V_1. First we assume that $\theta \circ f^*(\phi)$ is not identically zero. Then $\theta \circ f^*(\phi)$ is written locally as

$$(v_1, v_2, \dots, v_r) \longmapsto h_1(x)v_1 + h_2(x)v_2 + \cdots + h_r(x)v_r,$$

where $h_1, h_2, \dots, h_r$ are local holomorphic functions. If $h_1(x) = h_2(x) = \cdots = h_r(x) = 0$ at $x = p$, then $\theta \circ f^*(\phi)$ degenerates at p. Let n be the order of zero of $\theta \circ f^*(\phi)$ which is defined to be the minimum of the orders of zeros of h_i's at p and let $t(x)$ be a local coordinate around p. We put $h'_i(x) = h_i(x)t(x)^{-n}$. Then the following map

$$(v_1, v_2, \dots, v_r) \longmapsto h'_1(x)v_1 + h'_2(x)v_2 + \cdots + h'_r(x)v_r$$

does not degenerate at p. Let $\{p_i\}$ be the set of points where $\theta \circ f^*(\phi)$ degenerates and n_i be the order of zero at p_i. Let R be the divisor $\sum_i n_i p_i$. We modify the map $\theta \circ f^*(\phi)$ at each point p_i as above, then we have an exact sequence:

$$f^*(V_1) \ \to \ L \otimes [R]^{\otimes(-1)} \ \to \ 0.$$

Since V_1 is numerically semi-positive, the degree of $L \otimes [R]^{\otimes(-1)}$ is non-negative. So the degree of L is non-negative. If $\theta \circ f^*(\phi)$ is identically zero, then θ factors through $f^*(V_3)$. So similarly as above we can prove that the degree of L is non-negative.

§7. Dimension Formula for the Full Modular Group

In this section we present the dimension of $S_\mu(\Gamma_2(1))$ by a *Mathematica* function and the generating function of them. This result was announced in [T3]. But it is rather difficult to evaluate them from the expression in [T3].

Theorem 7.1. *Let $N \geq 3$ and μ the holomorphic representation of $GL(2, \mathbf{C})$ whose signature is $(2j + k, k)$. Then $\chi_{jk} := \sum_i (-1)^i \dim H^i(\widetilde{X}_2(N), \mathcal{O}(\widetilde{V}_\mu - D))^{G(N)}$ is given by the following Mathematica function:*

```
SiegelFull[j_,k_]:=Block[{a,lj,lk,ljk,x,y},

    mod[x_,y_]:=Mod[x,y]+1;

    a=(2*j+1)*(2*j+k-1)*(2*j+2*k-3)*(k-2)/5/27/128;
    a=a-(2*j+1)*(2*j+2*k-3)/9/32;
    a=a+(2*j+1)/48;
    (* contribution of φ1 *)
    (* contribution of φ15(r) *)
    (* contribution of φ22(1,r,t) *)
    (* contribution of φ25(1,r,s,t) *)

    lk={1,-1};
    a=a+(2*j+k-1)*(k-2)/9/128*lk[[mod[k,2]]];
```

```
a=a-(2*j+2*k-3)/3/64*lk[[mod[k,2]]];
a=a+lk[[mod[k,2]]]/32;
(* contribution of φ_2 *)
(* contribution of φ_16(r) *)
(* contribution of φ_23(4,r,t) *)

a=a+(2*j+k-1)*(k-2)/3/64*lk[[mod[k,2]]];
a=a-(6*j+6*k-7)/3/64*lk[[mod[k,2]]];
a=a-1/12*lk[[mod[k,2]]];
a=a+1/32*lk[[mod[k,2]]];
a=a+1/8*lk[[mod[k,2]]];
(* contribution of φ_3 *)
(* contribution of φ_17(r) *)
(* contribution of φ_22(3,r,t) *)
(* contribution of φ_24(4,r,t) *)
(* contribution of φ_25(i,r,s,t) (i=4,5,6) *)

lj={1,-1};
a=a+(2*j+2*k-3)/3/64*lj[[mod[j,2]]];
a=a-lj[[mod[j,2]]]/16;
(* contribution of φ_4 *)
(* contribution of φ_23(2,r,t) *)

a=a+(2*j+2*k-3)/128*lj[[mod[j,2]]];
a=a-lj[[mod[j,2]]]/16;
(* contribution of φ_5 *)
(* contribution of φ_24(2,r,t) *)

lj={1,0,-1};
a=a+(2*j+2*k-3)*lj[[mod[j,3]]]/54;
a=a-lj[[mod[j,3]]]/6;
(* contribution of φ_6 *)
(* contribution of φ_25(2,r,s,t) and φ_25(3,r,s,t) *)

ljk={{-2+k,1-2*j-k,2-k,-1+2*j+k},{2-k,1-2*j-k,-2+k,-1+2*j+k}};
a=a+ljk[[mod[j,2],mod[k,4]]]/96;
ljk={{-1,1,1,-1},{1,1,-1,-1}};
a=a+ljk[[mod[j,2],mod[k,4]]]/16;
(* contribution of φ_7(1) and φ_7(2) *)
```

(* contribution of $\varphi_{18}(1,r)$ and $\varphi_{18}(2,r)$ *)

```
a=a+ljk[[mod[j,2],mod[k,4]]]/16;
```

(* contribution of $\varphi_{19}(1,r)$ and $\varphi_{19}(2,r)$ *)

```
ljk={{-3+2*j+2*k,1-2*j-k,2-k},{1+2*j,-1-2*j,0},
      {-1+2*j+k,3-2*j-2*k,-2+k}};
a=a+ljk[[mod[j,3],mod[k,3]]]/216;
ljk={{-2,1,1},{0,0,0},{-1,2,-1}};
a=a+ljk[[mod[j,3],mod[k,3]]]/36;
```

(* contribution of $\varphi_8(1)$ and $\varphi_8(2)$ *)

(* contribution of $\varphi_{20}(1,r)$ and $\varphi_{20}(2,r)$ *)

```
ljk={{-1-2*j,1-2*j-k,2-k,1+2*j,-1+2*j+k,-2+k},
      {3-2*j-2*k,3-2*j-2*k,0,-3+2*j+2*k,-3+2*j+2*k,0},
      {1-2*j-k,-1-2*j,-2+k,-1+2*j+k,1+2*j,2-k}};
a=a+ljk[[mod[j,3],mod[k,6]]]/72;
ljk={{0,-1,-1,0,1,1},{-2,-2,0,2,2,0},{-1,0,1,1,0,-1}};
a=a-ljk[[mod[j,3],mod[k,6]]]/12;
```

(* contribution of $\varphi_8(3)$ and $\varphi_8(4)$ *)

(* contribution of $\varphi_{20}(3,r)$ and $\varphi_{20}(4,r)$ *)

```
ljk={{-1,1,0},{0,0,0},{0,1,-1}};
a=a+ljk[[mod[j,3],mod[k,3]]]/9;
```

(* contribution of $\varphi_{21}(1,r)$ and $\varphi_{21}(2,r)$ *)

```
ljk={1,-1};
a=a+(2*j+1)/128*ljk[[mod[j+k,2]]];
```

(* contribution of $\varphi_9(1)$ *)

```
ljk={{1,1,-1,-1},{-1,1,1,-1},{-1,-1,1,1},{1,-1,-1,1}};
a=a+ljk[[mod[j,4],mod[k,4]]]/16;
```

(* contribution of $\varphi_9(2)$ and $\varphi_9(3)$ *)

```
ljk={{0,-1,1},{-1,1,0},{1,0,-1}};
a=a+(2*j+1)*ljk[[mod[j,3],mod[k,3]]]/108;
```

(* contribution of $\varphi_{10}(1)$ and $\varphi_{10}(2)$ *)

```
ljk={{1,-1},{-2,2},{1,-1}};
a=a+ljk[[mod[j,3],mod[k,2]]]/108;
```

(* contribution of $\varphi_{10}(3)$ *)

```
ljk={{2,1,-1,-2,-1,1},{-1,1,2,1,-1,-2},{-1,-2,-1,1,2,1}};
a=a+ljk[[mod[j,3],mod[k,6]]]/27;
(* contribution of φ10(i) (i = 4,5,6,7) *)

ljk={{0,1,-1},{-1,1,0},{-1,0,1},{0,-1,1},{1,-1,0},{1,0,-1}};
a=a+ljk[[mod[j,6],mod[k,3]]]/12;
(* contribution of φ10(8) and φ10(9) *)

ljk={{1,0,0,0,-1,0,-1,0,0,0,1,0},{-1,0,0,0,1,0,1,0,0,0,-1,0}};
a=a+ljk[[mod[j,2],mod[k,12]]]/12;
ljk={{0,0,0,1,0,1,0,0,0,-1,0,-1},{0,-1,0,-1,0,0,0,1,0,1,0,0},
      {0,1,0,0,0,-1,0,-1,0,0,0,1}};
a=a-ljk[[mod[j,3],mod[k,12]]]/12;
(* contribution of φ11(i) (i = 1,2,3,4) *)

ljk={{1,-1},{-2,2},{1,-1}};
a=a+ljk[[mod[j,3],mod[k,2]]]/36;
(* contribution of φ12 *)

lj={1,-1,-1,1};
lk={1,-1};
a=a+lj[[mod[j,4]]]*lk[[mod[k,2]]]/16;
(* contribution of φ13 *)

ljk={{1,0,0,-1,0},{-1,1,0,0,0},{0,0,0,0,0},
      {0,0,0,1,-1},{0,-1,0,0,1}};
a=a+ljk[[mod[j,5],mod[k,5]]]/5;
(* contribution of φ14(i) (i = 1,2,3,4) *)

Return[a];

]
```

From the above theorem and the vanishing theorem (Theorem 6.1), we have the following

Corollary 7.2. *If $j = 0$ and $k \geq 4$ or if $j > 0$ and $k \geq 5$, then the dimension of $S_\mu(\Gamma_2(1))$ is equal to* `SiegelFull[j_,k_]`.

Table 7.3. *Let χ_{jk} be as in Theorem 7.1. Then the generating function*

$$\sum_{j,k=0}^{\infty} \chi_{jk}\, s^j t^k$$

of χ_{jk} $(k \geq 0)$ *is a rational function of* s *and* t *whose denominator is*

$$(1-s^3)(1-s^4)(1-s^5)(1-s^6)(1-t^4)(1-t^5)(1-t^6)(1-t^{12}).$$

Let $f(s,t)$ *be the numerator.* $f(s,t)$ *is of degree* 17 *with respect to* s *and of degree* 26 *with respect to* t. *The coefficients of* $s^j t^k$ $(0 \leq j \leq 17,\ 0 \leq k \leq 26)$ *are given by the following matrix:*

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -2 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & -1 & 0 & -3 & 0 & -1 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 1 & 1 & 4 & 2 & 3 & 0 & 1 & -3 & -1 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & -2 & -1 & -2 & -1 & -1 & 1 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & -1 & -2 & -1 & -2 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & -2 & -2 & -2 & -1 & -1 & 1 & 0 & 1 & 0 & 1 \\
-1 & -1 & 0 & 1 & 2 & 2 & 2 & 0 & -1 & -3 & -3 & -3 & -1 & 0 & 1 & 1 & 1 & 0 \\
-1 & 1 & 1 & 2 & 1 & 2 & -1 & -2 & -4 & -3 & -3 & -1 & 1 & 1 & 2 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & -3 & -3 & -3 & -2 & 0 & 1 & 3 & 1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 1 & 0 & 1 & -1 & -2 & -2 & -4 & -2 & -2 & 2 & 0 & 2 & 1 & 1 & -1 \\
1 & 2 & 1 & 1 & -2 & -3 & -6 & -5 & -4 & -1 & 2 & 2 & 5 & 1 & 2 & -1 & 0 & -2 \\
-1 & 0 & 0 & 1 & 1 & 0 & 1 & -3 & -2 & -4 & -2 & -3 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & -3 & -3 & -2 & -1 & 0 & 2 & 2 & 2 & 1 & 1 & -1 \\
-1 & -1 & 0 & 1 & 1 & 1 & 2 & 0 & -1 & -2 & -2 & -3 & -1 & 1 & 1 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 & 1 & 2 & 2 & 1 & 1 & -1 & 0 & -2 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & -1 & -1 & -2 & -1 & 1 & 2 & 3 & 3 & 3 & 0 & 0 & -1 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & -1 & -1 & -1 & 0
\end{matrix}$$

References

[A1] T. Arakawa, *Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions*, Manuscr. Math. **44** (1983), 155–185.

[A2] ———, *Special values of L-functions associated with the spaces of quadratic forms and the representation of* $Sp\,(2n, \mathbf{F}_p)$, Adv. Stud. Pure Math., vol. 15, Academic Press, Boston MA, 1989, pp. 99–169.

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth Compactification of Locally Symmetric Varieties (Lie Groups: History, Frontiers and Applications, Vol 4.)*, Math. Sci. Press, Brookline MA, 1975.

[AS] M. G. Atiyah and I. M. Singer, *The index of elliptic operator* III, Ann. of Math. **87** (1968), 540–608.

[Ba] W. L. Baily, Jr., *On Satake's compactification of* V_n, Amer. J. Math. **80** (1958), 348–364.

[BLS] H. Bass, M. Lazard and J.-P. Serre, *Sous-groupes d'indice fini dans* $SL(n, \mathbf{Z})$, Bull. Amer. Math. Soc. **70** (1964), 385–392.

[BH] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces* I, Amer. J. Math. **80** (1958), 458–538.

[F] T. Fujita, *On Kähler fiber space over curves*, J. Math. Soc. Japan **30** (1978), 779–794.

[Gd] R. Godement, *Generalites sur les formes modulaires,* I, II, Fonctions Automorphes, Sém. H. Cartan, vol. 10, 1957/1958, École Norm. Sup.

[Gt] E. Gottschling, *Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen*, Math. Ann. **169** (1967), 26–54.

[Gr1] P. A. Griffiths, *The extension problem in complex analysis* II*; Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), 366–446.

[Gr2] ———, *Hermitian differential geometry, Chern classes and positive vector bundles*, Global Analysis (Papers in Honor of K. Kodaira), Univ. of Tokyo Press, Tokyo, 1969, pp. 185–251.

[Ha1] K. Hashimoto, *The dimension of the spaces of cusp forms on Siegel upper half plane of degree two.* (I), J. Fac. Sci. Univ. Tokyo **30** (1983), 403–488.

[Ha2] ———, *Representations of the finite symplectic group* $Sp\,(4, \mathbf{F}_p)$ *in the space of Siegel modular forms*, Contemp. Math. **53** (1986), 253–276.

[Hi] F. Hirzebruch, *Elliptische Differentaloperatoren auf Mannigfaltigkeiten*, Arbeitsgemeinschaft für Forschung des Lades Nordrheinn-Westfalen **33** (1965), 583–608.

[IS] T. Ibukiyama and H. Saito, *On L-functions of ternary zero forms and exponential sums of Lee and Weintraub*, J. Number Theory **48** (1994), 252–257.

[Ig] J.-I. Igusa, *A desingularization problem in the theory of Siegel modular functions*, Math. Ann. **168** (1967), 228–260.

[Is] M. Ise, *Generalized automorphic forms and certain holomorphic vector bundles*, Amer. J. Math. **86** (1964), 70–108.

[Ka] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

[KO] S. Kobayashi and T. Ochiai, *On complex manifolds with positive tangent bundles*, J. Math. Soc. Japan **22** (1970), 499–525.

[Ko] K. Kodaira, *On a differential geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 1268–1273.

[L] R. P. Langlands, *The dimension of spaces of automorphic forms*, Amer. J. Math. **85** (1963), 99–125.

[LW] R. Lee and S. H. Weintraub, *On a generalization of a theorem of Erich Hecke*, Proc. Nat. Acad. Sci. U. S. A. **79** (1982), 7955–7957.

[MM] Y. Matsushima and S. Murakami, *On certain cohomology groups attached to hermitian symmetric spaces.* (I), Osaka J. Math. **2** (1965), 1–35.

[Me] J. Mennicke, *Zur Theorie der Siegelschen Modulgruppe*, Math. Ann. **143** (1961), 115–129.

[Mu] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272.

[Nk] S. Nakano, *On complex analytic vector bundles*, J. Math. Soc. Japan **7** (1955), 1–12.

[Nm] Y. Namikawa, *A new compactification of the Siegel space and degeneration of abelian varieties.* I, II, Math. Ann. **221** (1976), 97–141, 201–241.

[Sta] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. **20** (1956), 259–281.

[Sto] T. Satoh, *On certain vector valued Siegel modular forms of degree two*, Math. Ann. **274** (1986), 335–352.

[Shm] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[Shn] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya J. Math. **58** (1975), 83–126.

[Sr] B. Srinivasan, *The character of the finite symplectic group* $Sp\,(4, q)$, Trans. Amer. Math. Soc. **131** (1968), 488–525.

[T1] R. Tsushima, *A formula for the dimension of spaces of Siegel cusp forms of degree three*, Amer. J. Math. **102** (1980), 937–977.

[T2] ———, *On the spaces of Siegel cusp forms of degree two*, Amer. J. Math. **104** (1982), 843–885.

[T3] ———, *An explicit dimension formula for the spaces of generalized automorphic forms with respect to* $Sp\,(2, \mathbf{Z})$, Proc. Japan Acad. Ser. A **59** (1983), 139–142.

[T4] ———, *The space of Siegel cusp forms of degree two and the representation of* $Sp\,(2, \mathbf{F}_p)$, Proc. Japan Acad. Ser. A **60** (1984), 209–211.

[T5] ———, *Dimension formula for the spaces of Siegel cusp forms of half integral weight and degree two* (in preparation).

[V] E. Viehweg, *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8.

[Y] T. Yamazaki, *On Siegel modular forms of degree two*, Amer. J. Math. **98** (1973), 39–53.

[Z] S. Zucker, *Hodge theory with degenerating coefficients:* L^2 *cohomology in Poincaré metric*, Ann. of Math. **109** (1979), 415–476.

Corrections to Theorem 3.2

2) $\tau(\varphi_2, \Phi_2)$ should be $2^{-7}3^{-2}(-1)^k((k-2)(2j+k-1)N^6$
$$-6(2j+2k-3)N^5+36N^4)\prod(1-p^{-2})^2.$$

16) $\tau(\varphi_{16}(r), \Phi_{16})$ should be $2^{-5}3^{-1}(-1)^k\left(\dfrac{12-(2j+2k-3)N}{(1-\zeta^r)}\right)N^2\prod(1-p^{-2})$.

17) $\tau(\varphi_{17}(r), \Phi_{17})$ should be $(-1)^k \left(\frac{8-(2j+2k-3)N}{(1-\zeta^r)} + \frac{4}{(1-\zeta^r)^2} \right) N^2 \prod (1-p^{-2})$

$$\times \begin{cases} 2^{-5}, & if\ 2 \nmid N \\ 2^{-3}3^{-1}, & if\ 2 \mid N \end{cases}.$$

Corrections to [T2]

p.849, line 12 from the bottom: "$3\pi/3$" should read "$3\pi/2$".
p.862, line 12: "$i^*(\Delta(\Phi_3))$" should read "$\overline{\psi}^*(\Delta(\Phi_3))$".
p.877, line 1 from the bottom: "$\frac{z_2+mz_2+n}{(cz_1+d)u}$" should read "$\frac{z_2+mz_1+n}{(cz_1+d)u}$".
p.870, line 5: The (3,3) coefficient of the matrix shoud be 1.
p.871, line 5: "$2^{-3}3^{-1)^k}$" should read "$2^{-3}3^{-1}(-1)^k$".
p.872, line 8: "Lemma (4.)" should read "Lemma (4.9)".
p.877, line 7: "$rt \not\equiv 0 \mod \ell$" should read "$r \not\equiv 0,\ t \not\equiv 0 \mod \ell$".
p.877, line 9: "$\pm\zeta^{(r+t)/2}$" should read "$\pm\zeta^{(r+t)/2},\ -1$".
p.877, line 10: "$rt \not\equiv 0 \mod \ell$" should read "$r \not\equiv 0,\ t \not\equiv 0 \mod \ell$".
p.877, line 10: "$\zeta^r,\ \zeta^t$" should read "$\zeta^r,\ \zeta^t,\ -1$".
p.877, line 11: "$\pm\zeta^{(r+t)/2}$" should read "$\pm\zeta^{(r+t)/2},\ -1$".
p.877, line 12: "$rt \not\equiv 0 \mod \ell$" should read "$r \not\equiv 0,\ t \not\equiv 0 \mod \ell$".
p.877, line 12: "$\zeta^r,\ \zeta^t$" should read "$\zeta^r,\ \zeta^t,\ -1$".
p.877, line 13: "$(r+s)(s+t)s \not\equiv 0 \mod \ell$" should read "$r+s \not\equiv 0,\ s+t \not\equiv 0,\ s \not\equiv 0 \mod \ell$".
p.877, line 15: "$(r+2s+t)s \not\equiv 0 \mod \ell$" should read "$r+2s+t \not\equiv 0,\ s \not\equiv 0 \mod \ell$".
p.878, line 3: "$\tau(\phi_{22}(2,r,t),\Phi_{22})$" should read "$\tau(\phi_{22}(3,r,t),\Phi_{22})$".

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`http://www.meiji.ac.jp/severs/math/tsushima.html`[6]

[6]This is false. The correct one is http://www.math.meiji.ac.jp/˜tsushima