VI-7 Growth Estimate of Generalized Eigenfunctions of $\Delta$ on Two-Dimensional Manifolds

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Abstract

Let $M$ be a Riemannian manifold which is homeomorphic to the exterior of a disk in $\mathbb{R}^2$. We consider in this paper the growth property of the solution of $\Delta u + \lambda u = 0$ where $\Delta$ is the Laplace-Beltrami operator and $\lambda$ is a positive constant. We suppose that the metric of $M$ is expressed globally on $M$ as

$$ds^2 = a(r, \theta) dr^2 + 2b(r, \theta) \rho(r) dr d\theta + c(r, \theta) \rho(r)^2 d\theta^2$$

with a positive increasing function $\rho(r)$ which tends to infinity as $r \to \infty$ and with functions $a(r, \theta)$, $b(r, \theta)$ and $c(r, \theta)$. Under appropriate assumptions on $\rho(r)$ and the assumption that the metric is sufficiently close to $ds^2 = dr^2 + \rho(r)^2 d\theta^2$ as $r \to \infty$, we can obtain an estimate from below for the solution $u$. We do not assume anything about the sign of the curvature of the manifold. As to the method of analysis, we apply an abstract theorem due to K. Masuda. The present paper refines the author’s previous results in some direction.

Key Words: Manifold, Laplace-Beltrami, Eigenfunction, Growth-Order.

§ 1. Introduction and the result.

The growth property of the solutions of $\Delta u + \lambda u = 0$ for the Laplace-Beltrami operator $\Delta$ of noncompact manifolds has been investigated by many authors. Most of the works dealt with complete manifolds which possess negative or positive definite curvature (cf., for example, [4], [5], [6], [9] and [12]). In the present paper we also treat the same problem but we require neither the completeness nor the definiteness of the sign of the curvature. Moreover, the growth property of the solutions at the infinity is derived only from the behavior of the metric near the infinity.

We consider a Riemannian manifold $M$ which is homeomorphic to the exterior of a disk in $\mathbb{R}^2$, and whose metric is expressed with global coordinates $r$ and $\theta$ ($r < \infty$, $\theta \in S^1$) as

$$(1.1) \hspace{1cm} ds^2 = a(r, \theta) dr^2 + 2b(r, \theta) \rho(r) dr d\theta + c(r, \theta) \rho(r)^2 d\theta^2$$

Here, $\rho$ is a positive increasing $C^2$-function of $r$ with $\rho \to \infty$ as $r \to \infty$ and $a, b$ and $c$ are $C^2$-functions of $r$ and $\theta$. Our objective is to find a growth property as $r \to \infty$ of the solutions of $\Delta u + \lambda u = 0$, where $\Delta$ is the Laplace-Beltrami operator of $M$ and $\lambda$ is an arbitrarily chosen positive constant.

We begin with assuming several properties of $\rho$. In what follows, $\dot{\rho} = d\rho/dr$ and we write $\rho^{-1}$ and $\dot{\rho}^{-1}$ for $1/\rho$ and $1/\dot{\rho}$, respectively.

**ASSUMPTION 1.**  
(i) $\dot{\rho}(r) > 0$ \hspace{0.5cm} (a. e. $r$), \hspace{0.5cm} $\rho(r) \to \infty$ \hspace{0.5cm} ($r \to \infty$).

(ii) $\dot{\rho} = o(\rho) \hspace{0.5cm} (r \to \infty)$.

(iii) $\ddot{\rho} = o(\dot{\rho}) \hspace{0.5cm} (r \to \infty)$.

(The same properties were required in the author’s previous work [7] which treated rotationally symmetric manifolds having the metric $ds^2 = dr^2 + \rho(r)^2 d\omega^2$, $\omega \in S^{n-1}$.)

Next, we make the following conditions on the functions $a$, $b$ and $c$. (Here and in the sequel, the suffixes like $\alpha$, $\gamma$ represent partial derivatives. Moreover, we simply write $f = o(g)$ to mean that $\lim_{r \to \infty} f(r, \theta)/g(r) = 0$ holds uniformly in $\theta$. Similarly, $f = O(g)$ means that $f(r, \theta)/g(r)$ is bounded uniformly in $\theta$ as $r \to \infty$.)

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ASSUMPTION 2. (i) \( a = 1 + o(1) \), \( c = 1 + o(1) \), \( b = o(1) \).
(ii) \( a_\rho = o(\rho^{-1}) \), \( c_\rho = o(\rho^{-1}) \), \( b_{\rho r} = o(\rho^{-1}) \), \( bb_{\rho} = o(\rho^{-1}) \), \( a_\rho = o(\rho) \), \( c_{\rho} = o(\rho) \), \( b_{\rho} = o(\rho) \).

\[ \sup_{z \in \mathbb{R}^1} |a_{\rho}| \in L^1(\mathbb{R}_0, \infty), \sup_{z \in \mathbb{R}^1} |c_{\rho}| \in L^1(\mathbb{R}_0, \infty), \sup_{z \in \mathbb{R}^1} |bb_{\rho}| \in L^1(\mathbb{R}_0, \infty), \]
\[ \rho^{-1} \sup_{z \in \mathbb{R}^1} |b_{\rho}| \in L^1(\mathbb{R}_0, \infty), \]
\[ \rho^{-1} \sup_{z \in \mathbb{R}^1} |b_{\rho}c_{\rho}| \in L^1(\mathbb{R}_0, \infty), \]
\[ \rho^{-2} \sup_{z \in \mathbb{R}^1} |b_{\rho}| \in L^1(\mathbb{R}_0, \infty), \]
\[ a_{\rho} = o(\rho^{-1}) \), \( c_{\rho} = o(\rho^{-1}) \), \( b_{\rho r} = O(\rho^{-1}) \), \( a_\rho = o(\rho) \), \( c_{\rho} = o(\rho) \), \( b_{\rho} = O(\rho) \), \( a_{\rho} = o(\rho) \), \( c_{\rho} = o(\rho) \), \( b_{\rho} = o(\rho) \).

Here we note that the Laplace-Beltrami operator \( \Delta \) is explicitly given by

\[ (1.2) \quad \Delta = \frac{1}{\rho h} \left\{ \partial_r \left( \frac{\partial}{\partial r} (\rho \partial_r) \right) - \partial_r \left( \frac{b}{h} \partial_r \right) - \partial_r \left( \frac{b}{h} \partial_r \right) + \frac{1}{\rho} \partial_r \left( \frac{a}{h} \partial_r \right) \right\} \]

where \( h = h(\rho, \theta) \) is the function defined by

\[ h = \sqrt{ac - b^2} \]

Then we can state our principal result as follows.

**THEOREM 1.1.** Let \( \lambda \) be an arbitrary positive constant and \( u \) a solution of the equation

\[ (1.3) \quad \Delta u + \lambda u = 0 \]

which does not vanish identically. Suppose that Assumptions 1 and 2 are satisfied. Then for every positive number \( k \) one can find an \( r_1(\geq r_0) \) and positive constants \( C \) and \( C' \) such that

\[ (1.4) \quad \int_{\{r \leq r_1, r \leq \rho \leq R \}} |u|^2 dM \geq \int_{r_1}^R \frac{dr}{\rho^2} - C' \left( \log R - r_1 + 2 \right) \]

holds true (\( dM \) being the measure corresponding to the metric).

**COROLLARY.** Under the same condition of the theorem, (1.3) can not have any nontrivial \( L^2 \)-solution if

\[ \int_{r_1}^\infty \frac{dr}{\rho^2} = \infty \]

holds for some positive constant \( k \).

In [8] the author treated the same problem. But [8] aimed at finding less smooth \( \rho(r) \) which guarantees (1.4). In fact, the theorem of [8] admits a function \( \rho \) which is not of class \( C^1 \). In exchange for that, however, we had to assume in [8] rapid decay of \( a(r, \theta) = 1, b(r, \theta), c(r, \theta) = 1 \) and their derivatives. On the contrary, the present paper requires that \( \rho \) is \( C^2 \) and satisfies Assumption 1, but \( a = 1, b \) and \( c = 1 \) can decay more slowly than those in [8].

Tayoshi [13] treated the spectra of second order elliptic operators on \( n \)-dimensional noncompact manifolds. His theorem is not restricted to the Laplace-Beltrami operator, but as far as the manifolds with the metric of the form (1.1) is concerned, our result admits a wider class of \( \rho \)'s. For example, \( \rho = \log r \) satisfies Assumption 1 of the present paper, while it does not fit Tayoshi's theorem.


Although there could be various methods to prove Theorem 1.1, here we make use of k. Masuda's theorem which enables us to organize a concise proof.

In [10] Masuda treated a growth property of vector-valued functions satisfying certain second order differential equations. Later he extended his result to the case of differential inequalities (11). Although the result of [10] was absorbed in that of [11], the techniques of [10] and [11] are different to some extent. And in the present paper, we shall use the original method [10] by making a new modification. The essential point of our alteration is that we put an extra Assumption 9 below in order to get the conclusion simply in a definite form.

Since Masuda has not published [10], we shall describe here Masuda's theorem together with its full proof given by
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him. As mentioned above, however, a part of the proof is modified by the author in order to get Proposition 2.2.

Throughout this section $H$ is a Hilbert space, $D$ is a subspace of $H$ (not necessarily dense), $\langle, \rangle$ and $(\cdot, \cdot)$ denote the norm and inner product of $H$, respectively. $A_1(r)$, $A_2(r)$, $B_1(r)$ and $B_2(r)$ are linear operators on $H$ depending on a parameter $r$ which ranges over the interval $(r_1, \infty)$. For each fixed $r$ they assume values in $H$ and their domains are supposed to include $D$. Let $v(r)$ $(r_1 \leq r < \infty)$ be a function of $r$ with values in $D$ which satisfies the differential equation

$$\left( \frac{d}{dr} + B_1(r) \right)^2 v(r) + B_2(r) \frac{d}{dr} v(r) + A_1(r) v(r) + A_2(r) v(r) = 0$$

where the differentiation is in the strong sense. We here introduce several assumptions.

**ASSUMPTION 3.** For arbitrary elements $x$ and $y$ of $D$ and for each $r \in (r_1, \infty)$,

$$(A_1(r)x, y) = (x, A_1(r)y).$$

**ASSUMPTION 4.** $B_i(r)$ is decomposed into a sum of linear operators as $B_i(r) = B_{i\alpha}(r) + B_{i\beta}(r)$ where

$$B_{i\alpha}(r)x, y = (x, B_{i\alpha}y) \quad (x, y \in \text{domain of } B_{i\alpha}),$$

$$B_{i\beta}(r)x, y = - (x, B_{i\beta}y) \quad (x, y \in \text{domain of } B_{i\beta}).$$

Moreover, $D$ is included in each domain of the operators $B_{i\alpha}(r)B_{i\alpha}(r)$, $B_{i\alpha}(r)B_{i\beta}(r)$, $B_{i\beta}(r)B_{i\alpha}(r)$, $B_{i\beta}(r)B_{i\beta}(r)$.

**ASSUMPTION 5.** $B_i(r)$ is a bounded operator and

$$\beta(r) := \|B_i(r)\|$$

is a continuous function of $r$ and belongs to $L^1(r_1, \infty)$.

**ASSUMPTION 6.** For every fixed $x$ in $D$, $A_i(\cdot)x$, $B_{i\alpha}(\cdot)x$ and $B_{i\beta}(\cdot)x$ belong to $C^1((r_1, \infty); H)$ with respect to the norm of $H$. Also $B_i(\cdot)x$ and $A_i(\cdot)x$ belong to $C((r_1, \infty); H)$.

In what follows differentiation with respect to $r$ is denoted by a dot.

**ASSUMPTION 7.** There exist nonnegative functions $\varphi(r)$, $\psi(r)$ and $\eta(r)$ (where $\int_{r_1}^\infty \psi(r)dr = \infty$) such that

$$\varphi(r)^{-1}\varphi(r) \left( \|y\|^2 + \|[[A_1(r) + \eta(r)]x, x]\| - 2Re(\{B_1(r) + B_2(r)\}y, y) \right)$$

$$+ 2Re(\{\eta(r) + B_2(r)A_1(r) - A_2(r)\}x, y) - 2Re(\{A_1(r) + \eta(r)\}x, B_1(r)x) + \|\tilde{A}(r) + \tilde{\eta}(r)\|x\|^2$$

holds for each $x \in D$ and $y \in \cap_{r > r_1} D(B_i(r))$. ($D(\cdot)$ denotes the domain of definition.)

**ASSUMPTION 8.** For every $x \in D$ one has

$$\eta(r)\|x\|^2 + 2Re(B_{i\alpha}(r)x, B_{i\alpha}(r)x) - Re(B_1(r)x, x) + Re(B_2(r)B_{i\alpha}(r)x, x) - Re(A_3(r)x, x) \geq \frac{1}{4}\beta(r)^2\|x\|^2.$$  

**DEFINITION 1.** Let $K$ be an arbitrary positive constant. If $\int_{r_1}^\infty \psi(s) e^{-Ks} ds$ is finite, we set

$$\xi_K(R) := \int_{r_1}^R \psi(s) e^{-K(s-r)} ds dr.$$

**ASSUMPTION 9.** For each $K > 0$ we have either

$$\int_{r_1}^\infty \psi(s) e^{-Ks} ds = \infty$$

or

$$\lim_{r \to \infty} \xi_K(K) = \infty.$$

**THEOREM 2.1.** Suppose that Assumptions 3 – 9 are satisfied. If $v(\cdot) \in C^2((r_1, \infty); D)$ is a solution of (2.1) in the sense of the strong differentiation, then with a positive constant $C = C_0$ and a number $r_2 \geq r_1$ we have

$$\|\phi(r) + B_1(r)v(r)\|^2 + (A_1(r)v(r), v(r)) + \eta(r)\|v(r)\|^2 \geq C\varphi(r)^{-1} \quad (r \geq r_2).$$

(67)
provided that \( v \) does not vanish identically.

PROOF. Let \( F \) be a real-valued function of \( r \) defined by

\[
F(r) = \|\dot{v}(r) + B_1(r)v(r)\|^2 + (A_1(r)v(r), v(r)) + \eta(r)\|v(r)\|^2.
\]

In what follows, we simply write \( v, A_1 \) and \( \overrightarrow{\phi} \) for \( v(r), A_1(r) \) and \( 1/\phi(r) \), respectively.

At first we show

**PROPOSITION 2.1.** Under Assumptions 3–7, we have

\[
\frac{d}{dr}(\overrightarrow{\phi} F) \geq \psi \|v\|^2 \quad (r > r_1).
\]

PROOF. In virtue of Assumptions 3, 4 and 6, a straightforward calculation leads to

\[
\dot{F} = 2\Re\left( \frac{d}{dr}(\dot{v} + B_1 v), \dot{v} + B_1 v \right) + 2\Re\langle A_1 v, \dot{v} \rangle + (A_1, v) + \eta \Re(v, \dot{v} + B_1 v)
\]

\[
= 2\Re\left( \left( \frac{d}{dr} + B_1 \right)^2 v, \dot{v} + B_1 v \right) - 2\Re\langle B_1(\dot{v} + B_1 v), \dot{v} + B_1 v \rangle
\]

\[
+ 2\Re\langle A_1 v, \dot{v} + B_1 v \rangle - 2\Re\langle (A_1 + \eta) v, (A_1 + \eta) v \rangle
\]

\[
+ 2\Re\langle \eta - A_1 v, \dot{v} + B_1 v \rangle - 2\Re\langle (A_1 + \eta) v, B_1 v \rangle + ((A_1 + \eta) v, v).
\]

Therefore, by setting \( x = v, y = \dot{v} + B_1 v \) and from Assumption 7 we have

\[
\overrightarrow{\phi} \frac{d}{dr}(\overrightarrow{\phi} F) = \overrightarrow{\phi} \left\{ \|v\|^2 + \Re\langle (A_1 + \eta) v, x \rangle \right\} - 2\Re\langle B_1 + B_3 \rangle y, y \rangle
\]

\[
+ 2\Re\langle B_1 B_3 x, y \rangle + 2\Re\langle (\eta - A_1) x, y \rangle - 2\Re\langle (A_1 + \eta) x, B_1 x \rangle + ((A_1 + \eta) x, x)
\]

\[
\geq \overrightarrow{\phi} \|v\|^2.
\]

The following proposition itself is not contained in Masuda's papers. However, the crucial part of our proof follows Masuda's argument which was used to prove the second inequality of his theorem (Theorem A included in the appendix). For the rest of our proof, we make use of Agmon's idea in [1].

**PROPOSITION 2.2.** Under Assumptions 3–9, there exists a number \( r_2 \) such that \( F(r_2) > 0 \).

PROOF. To show the proposition by a contradiction argument, we assume \( F(r) \leq 0 \) for \( r \geq r_1 \). From Proposition 2.1 we see

\[
-\overrightarrow{\phi}(r) F(r) = -\overrightarrow{\phi}(t) F(t) + \int_{r_1}^{t} \frac{d}{ds}(\overrightarrow{\phi}(s) F(s)) \, ds
\]

\[
\geq \int_{r_1}^{t} \psi(s) \|v(s)\|^2 \, ds.
\]

The last term above is a monotone increasing function of \( t \), while the first one does not depend on \( t \). Therefore, letting \( t \to \infty \) we get

\[
\overrightarrow{\phi}(r) F(r) \geq \int_{r_1}^{\infty} \psi(s) \|v(s)\|^2 \, ds
\]

together with the convergence of the right-hand integral.

Now, from the hypothesis that \( v \) is continuous and does not vanish identically, we can find a subinterval \( I \) of \( (r_1, \infty) \) in which \( v(r) \) never vanishes. For each \( r \in I \) set

\[
f(r) = \log \|v(r)\|^2.
\]

Then we note that

\[
f = \frac{2}{\|v\|^2} \{ \Re(\dot{v}, v) + \|v\|^2 - 2\Re(\dot{v}, v) \}.
\]

From the equation (2.1) we get

(68)
In virtue of (2.8), Assumption 8 leads

\[ \mathcal{J}(r) \geq \frac{2}{\|v\|^2} \left( \frac{1}{\varphi r} \int_{r}^{\infty} \psi(s) \|v(s)\|^2 ds + \frac{1}{4} \beta \|v\|^2 \right) \]

since \( |B_s|^2 = \beta \) and \( \text{Re}(B_s v, v) = 0 \). We now use the following proposition, whose proof will be given below.

**PROPOSITION 2.3.** For any two elements \( x, y \) of \( H (x \neq 0) \) we have

\[ \frac{1}{|x|^2} \left[ |y|^2 - \frac{\beta}{2} |x||y| + \frac{\beta^2}{8} |x|^2 - \frac{\|\text{Re}(x, y)\|^2}{|x|^2} \right] \geq -\frac{\beta}{2} \left| \text{Re}(x, y) \right|. \]

Applying this proposition to the right-hand side of (2.10) we see

\[ \mathcal{J}(r) \geq \frac{2}{\|v\|^2} \int_{r}^{\infty} \psi(s) \|v(s)\|^2 ds - 2 \beta \cdot \frac{\|\text{Re}(\dot{v} + B_s v, v)\|^2}{\|v\|^2}. \]

But since

\[ \mathcal{J}(r) = \frac{2 \text{Re}(\dot{v}, v)}{|v|^2} - \frac{2 \text{Re}(\dot{v} + B_s v, v)}{|v|^2}, \]

(2.11) gives rise to

\[ \int_{r}^{\infty} \psi(s) \|v(s)\|^2 ds = \frac{2}{\varphi(r)} \int_{r}^{\infty} \psi(s) e^{\sigma(s - r)} ds, \]

where \( \sigma(r) := \text{sgn} \, \mathcal{J}(r) \).

Now we turn to follow Agmon’s idea [1]. From (2.12) we obtain, in particular, that

\[ \mathcal{J}(r) + \beta(r) \sigma(r) \mathcal{J}(r) = \exp \left[ -\int_{r}^{\infty} \beta(t) \sigma(t) dt \right] \frac{d}{dr} \left( \exp \left[ \int_{r}^{\infty} \beta(t) \sigma(t) dt \right] \mathcal{J}(r) \right) \geq 0, \]

and hence there exists a positive constant \( K \) for which

\[ \mathcal{J}(r) \geq -K \]

holds, because \( \beta \in L^1_1 (r, \infty) \). Accordingly, one has

\[ f(s) - f(r) \geq -K (s-r) \quad (r < s, \ r, s \in I). \]

So far, we have supposed that \( r, s \in I \). But seeing (2.13), we come to know that \( f(s) \) can by no means go to \(-\infty\) at any finite point. In other words, \( \|v(r)\| \) must be strictly positive in \( r \leq r < \infty \). This fact implies also that (2.13) holds for \( r \leq r < s < \infty \).

Thus, in particular, if \( \int_{r}^{\infty} \psi(s) e^{-\beta(s)} ds = \infty \), it leads to a contradiction because the right-hand side of (2.12) is finite. Hence the \( \zeta_k (R) \) of Definition 1 exists. Substituting (2.13) into (2.12) and repeating the same argument, we have
\[
\exp \left( \int_{1}^{2} \beta(t) \psi(t) \, dt \right) f(R) \geq \text{const.} + \int_{1}^{2} \frac{2}{\varphi(r)} \exp \left( \int_{r}^{2} \beta(t) \psi(t) \, dt \right) \int_{r}^{\infty} \psi(s) e^{-\sqrt{s(r-r')}} \, ds \, dr
\]
\[
\geq \text{const.} + 2 \exp \left( - \int_{1}^{2} \beta(t) \psi(t) \right) \psi(R).
\]

Therefore, by virtue of Assumption 9, \( f(R) \rightarrow \infty \) and hence \( \| \nu(R) \| \rightarrow \infty \) as \( R \rightarrow \infty \). But this fact is incompatible with \( \int_{r}^{\infty} \psi(s) \| \nu(s) \|^2 \, ds < \infty \) and \( \int_{1}^{\infty} \psi(s) \, ds = \infty \) which follow from (2.8) and Assumption 7, respectively. Thus Proposition 2.2 has been established.

From Propositions 2.1 and 2.2 we see that \( \varphi(r) F(r) \geq \varphi(r_{2}) F(r_{2}) \) \( (r \geq r_{2}) \), which is nothing but Theorem 2.1.

**Proof of Proposition 2.3** (by Masuda). Set \( y = (p + iq)x + z, \ p, \ q \in \mathbb{R}, \ x, \ z \in \mathbb{H}, \ z \perp x, \) and \( \| z \| = \xi x \| \). Then
\[
\frac{1}{\| x \|^2} \left( \| y \|^2 - \frac{\beta}{2} \| x \|^2 - \frac{1}{2} \left( \text{Re} (x, y) \right)^2 \right)
= p^2 + q^2 + |\xi|^2 - \frac{\beta}{2} (p^2 + q^2 + |\xi|^2 - p^2)
\geq |q|^2 + |\xi|^2 - \frac{\beta}{2} (|p| + |q| + |\xi|)
\geq - \frac{\beta}{8} \frac{|p|^2}{|p|}.
\]

\( \Box \)

§ 3. **Proof of Theorem 1.1.**

The proof of Theorem 1.1 is achieved by reducing the equation (1.3) to an equation of the type (2.1). We note that a function \( u(r, \theta) \) of the variables \( r \) and \( \theta \) can be regarded as a vector-valued function of \( r \) which admits the values in \( L^2(S^1) \).

For convenience, we write the explicit form of the Laplace-Beltrami operator again,

\[
\Delta = \frac{1}{\rho^2} \left( \partial_r \left( \frac{\rho \partial_r}{\rho} \right) - \frac{1}{\rho^2} \partial_r \left( \frac{b}{\rho} \right) - \partial_r \left( \frac{b}{\rho} \right) + \frac{1}{\rho} \partial_r \left( \frac{\xi}{\rho} \right) \right)
\]

where \( \rho = h(r, \theta) = \sqrt{a^2 - b^2} \). For the \( C^1 \)-solution of the equation \( \Delta u + \lambda u = 0 \) which we are considering, we put
\[
v = \rho \rho h u.
\]

Then \( v \) satisfies
\[
(\partial_r \xi, \xi) v + \mu \partial_r v + \rho^{-1} \partial_r (\nu \partial_r) v + q v + \xi \partial_r v + \xi v = 0,
\]

where \( \kappa, \mu, v, q, \xi \) and \( \xi \) are given by

\[
\begin{align*}
\kappa &= \rho^{-1} c^{-1} b, \\
\mu &= 2 \rho^{-1} k, h c^{-1} b - 2h^{-1} h c^{-1} b + \rho^{-1} c^{-1} b + c^{-1} c, \\
v &= c^{-1} a - c^{-2} b^2, \\
q &= \lambda h c^{-1} = \lambda (a - c^{-1} b), \\
\xi &= 2 \rho^{-1} h^{-1} h c^{-1} b - 2 \rho^{-1} h^{-1} h c^{-1} a - \rho^{-1} c^{-1} c, b - \rho^{-1} c^{-1} c b^2 \\
&+ \rho^{-1} c^{-2} b b + \rho^{-1} c^{-2} c a, \\
(3.3)
\xi &= \frac{1}{4} \rho^{-1} \frac{\rho^{-1} \rho}{2} + \frac{1}{2} \rho^{-1} \rho + \frac{1}{2} \rho^{-1} \rho \ c^{-1} b + \rho^{-1} \rho h^{-1} h c^{-1} b - \frac{1}{2} \rho^{-1} \rho h^{-1} h c^{-1} b \\
&+ \frac{5}{4} \rho^{-1} \rho h^{-1} h c^{-1} a - \frac{1}{2} \rho^{-1} \rho h^{-1} h c^{-1} a + \frac{1}{2} \rho^{-1} \rho h^{-1} h c^{-1} b - \frac{1}{2} \rho^{-1} \rho c^{-1} c r \\
&+ \frac{1}{2} \rho^{-1} \rho h^{-1} h + \frac{1}{2} \rho^{-1} \rho h^{-1} h c^{-1} b - \frac{5}{2} \rho^{-1} \rho h^{-1} h c^{-1} b \\
&- \frac{1}{2} h^{-1} h c^{-1} c + \frac{5}{4} \rho^{-1} \rho h^{-1} h + \rho^{-1} h^{-1} h c^{-1} b - \frac{1}{2} h^{-1} h r r.
\end{align*}
\]
Hence, by putting

\[
H = L^2(S^n), \quad D = H^2(S^n), \\
B_r = -\kappa \partial_r, \quad B_s = \frac{1}{2} \kappa s, \quad B_\phi = -\kappa \partial_\phi - \frac{1}{2} \kappa \kappa, \quad D(B(r)) = D,
\]

(3.4)

one has the equation (2.1) for \( v(r) = v(r, \cdot) \), where we regard \( v(r, \theta) \) is a vector-valued function of \( r \) with values in \( D \).

From Assumptions 1 and 2 we see

\[
\begin{align*}
 h = 1 + o(1), & \quad h_x = o(\rho^{-1} \rho) , \quad h_\theta = o(\delta), \\
h_r = o(\rho^{-1} \rho), & \quad h_\theta = o(\delta), \quad h_\phi = o(\delta), \\
\kappa = o(\rho^{-1}), & \quad \mu = o(\rho^{-1} \rho), \quad \nu = 1 + o(1), \quad \eta = o(1), \\
\xi = o(\rho^{-1} \rho), & \quad \zeta = o(\rho^{-1} \rho), \quad \nu = o(\rho), \quad \eta = o(\rho), \\
\kappa = o(\rho^{-1} \rho), & \quad \mu = O(\rho), \quad \nu = o(\rho), \quad \eta = o(\rho), \\
\kappa = o(\rho^{-1} \rho), & \quad \mu = O(1), \quad \nu = o(\rho^{-1} \rho), \quad \eta = o(\rho^{-1} \rho), \\
\sup \| \mu \| \in L^1(\nu, \infty).
\end{align*}
\]

(3.5)

We examine Assumption 7 with \( \phi(r) = \rho(r)^k \) and \( \eta(r) = \rho(r)^{-1} \rho(r) \). Here \( k \) is an arbitrary number satisfying \( 0 < k < 2 \), and \( \epsilon \) is a small number determined later. As a matter of convenience, let us denote the left-hand side of (2.2) by \( \Gamma[x, y] \) so that (2.2) is written as \( \Gamma[x, y] \geq \psi(r) \| x \|^2 \). From (3.4) we have

\[
\Gamma[x, y] = k \rho^{-1} \rho \{ \| y \|^2 + (\rho^{-1}(\omega x) + \omega x + \rho^{-1} \rho x, x) \} - 2 \Re \{ (k \omega + \mu y, y) + 2 \Re (\rho^{-1} \rho x - \mu \omega x - \xi \omega - \zeta \omega, y) \\
- 2 \Re (x \omega x, y) - 2 \Re (\xi \omega, y) - 2 \Re (\zeta \omega, y) + \rho^{-1} \rho [\| x \|^2] - (x \omega y, y) - 2 (\mu y, y) + 2 \rho^{-1} \rho \Re (x, y) \\
+ 2 \rho^{-1} \rho (x \omega x, x) - \rho^{-1} \rho (\omega x, x) + (q, x, x) + \epsilon (\rho^{-1} \rho^2 + \rho^{-1} \rho) \| x \|^2.
\]

In the calculation above we have made use of the relation \( 2 \Re (x \omega, x) = - (x \omega, x) \). Further, the Schwarz inequality and application of the well-known inequality regarding arithmetic and geometric means with an appropriate weight yield

\[
\Gamma[x, y] \geq \rho^{-1} \rho \{ \| y \|^2 + \rho^{-1} \rho \| x \|^2 \} - 2 \Re \{ (k \omega + \mu y, y) + 2 \Re (\rho^{-1} \rho x - \mu \omega x - \xi \omega - \zeta \omega, y) \\
- 2 \Re (x \omega x, y) - 2 \Re (\xi \omega, y) - 2 \Re (\zeta \omega, y) + \rho^{-1} \rho [\| x \|^2] - (x \omega y, y) - 2 (\mu y, y) + 2 \rho^{-1} \rho \Re (x, y) \\
+ 2 \rho^{-1} \rho (x \omega x, x) - \rho^{-1} \rho (\omega x, x) + (q, x, x) + \epsilon (\rho^{-1} \rho^2 + \rho^{-1} \rho) \| x \|^2
\]

\supbeing \sup_{w \in \nu}. Considering (3.5) and (ii) of Assumption 1, we conclude that if \( r_1 \) is taken sufficiently large and \( r \geq r_1 \), then both of the coefficients of \( \| y \|^2 \) and \( \| x \|^2 \) are nonnegative while the coefficient of \( \| x \|^2 \) is greater than or equal to \( \frac{1}{2} \kappa \lambda - \epsilon \) \( \rho^{-1} \rho \). Therefore, if we choose \( \epsilon < \kappa \lambda / 4 \), then

\[
\Gamma[x, y] \geq \varphi(r)^{-1} \psi(r) \| x \|^2
\]

holds true, where
Obviously $\psi$ is not integrable. Thus Assumption 7 has been ascertained.

Substituting (3.4) into (2.3) and considering (3.5), we see that

$$
\eta \|x\|^2 + 2 \Re(B_{1\alpha}, B_{1\alpha}x) - \Re(B_{1\alpha}x, x) + \Re(B_{1\alpha}B_{1\alpha}x, x) - \Re(A_{1\alpha}x, x)
$$

$$
= e^{-\nu t} \eta \|x\|^2 + 2(\nu - \frac{1}{2} \nu^2) \Re(A_{1\alpha}x, x)
$$

$$
+ \Re(-\mu \nu, x) - \Re(\xi_0 + \xi, x)
$$

$$
= e^{-\nu t} \eta \|x\|^2 + \frac{1}{2} (\mu \nu + \xi_0 + \xi, x) + (\xi_0 + \xi, x)
$$

$$
\geq \frac{1}{2} e^{-\nu t} \eta \|x\|^2
$$

$$
\geq \frac{1}{4} \|\mathcal{A}(\mu(r, \theta))\|^2 \eta \|x\|^2
$$

for $r \geq r_1$, taking a larger $r_1$ if necessary. But this implies Assumption 8, because

$$
\beta(r) = \sup_{\nu} \|\mathcal{A}(\mu(r, \theta))\| \in L^1(r_1, \infty).
$$

The expression of $\xi_\nu(R)$ of Definition 1 is now in order.

$$
\xi_\nu(R) = \int_{r_1}^{r} \rho(r) \int_{r_1}^{\infty} \frac{1}{4} e^{-\nu t} \xi_\nu(s) e^{-\nu (s-r)} ds dr.
$$

Since $\rho(r) \rightarrow 0$ is monotone decreasing, a change of the order of integration leads to

$$
\lim_{K \rightarrow -\infty} \xi_\nu(R) = \frac{k}{4} \int_{r_1}^{\infty} \rho(s) \int_{r_1}^{r} e^{-\nu t} \xi_\nu(s) e^{-\nu (s-r)} ds dr
$$

$$
\geq \frac{k}{4} \int_{r_1}^{\infty} \rho(s) \int_{r_1}^{r} e^{-\nu t} \xi_\nu(s) e^{-\nu (s-r)} ds dr
$$

$$
\geq \frac{k}{4} \int_{r_1}^{\infty} (1 - e^{-\nu t}) ds
$$

which verifies Assumption 9.

We thus have a number $r_1$ for which Assumptions 7–9 hold. Since the solutions have a unique continuation property (see e.g. Aronszajn [3]), the hypothesis $u \equiv 0$ on $M$ implies $\nu(r) \equiv 0$ for $r \geq r_1$. So we can eventually apply Theorem 2.1 and have

$$
F(r) \geq C \rho(r)^{-1} \quad (r \geq r_1)
$$

as a consequence of (2.5). Using the symbol

$$
D = \partial_{r} - \partial_{\alpha}
$$

and substituting $\nu(r) = \rho(r)^{-1}$, we see that

$$
D \nu = \int_{f} \left( |Dv|^2 + \rho^{-2} \nu \omega \right) + \left( q + \eta \right) \nu |v|^2 d\theta \geq C \rho(r)^{-2} \quad (r \geq r_1).
$$

From this expression and the equation (3.2), one is now able to derive the desired estimate for $u$. To show it, we begin with several formulae. For brevity, set

$$
\int_{f} = \int_{r_1}^{r} \int_{f} \frac{d\theta d\alpha}{d
}.$$
PROPOSITION 3.1. (i) For any $C^1$-function $f(r)$ satisfying $f(r_i) = f(R) = 0$ and any $C^1$-function $g(r, \theta)$, one has

\[
\int \int Df \cdot g = - \int \int f \cdot Dg + \int \int \kappa g
\]

\[
2 \Re \int \int f \cdot Dg \cdot \bar{g} = \int \int (\kappa \theta - f)^2 |g|^2
\]

(ii) For any $C^1$-function $f(r)$ such that $f(r_i) = f(R) = 0$ and any $g(r, \theta)$ of class $C^2$, the relation

\[
\int \int D^2 f \cdot g = \int \int f \cdot D^2 g + \int \int \kappa Df \cdot g - \int \int \kappa f \cdot Dg
\]

holds.

Now, let $\sigma_\theta(r)$ $(r_1 \leq r < \infty, r_1 + 2 \leq R)$ be a family of $C^\infty$-functions satisfying

\[
0 \leq \sigma_\theta(r) \leq 1 \quad (r_1 \leq r < \infty),
\]

\[
\sigma_\theta(r_1) = \sigma_\theta(r_2) = 0, \quad \sigma_\theta(0) = 0 \quad (R \leq r < \infty),
\]

\[
\sigma_\theta(r) = 1 \quad (r_1 + 1 \leq r \leq R - 1),
\]

such that $|\sigma_\theta(r)|$ and $|\sigma_\theta'(r)|$ and $|\sigma_\theta''(r)|$ are bounded by a constant which is independent of $R$. If $v$ is a solution of (3.2) then

\[
\int \int \sigma_\theta |v|^2 = \int \int D^2 \sigma_\theta |v|^2
\]

\[
= \int \int \sigma_\theta D^2 |v|^2 + \int \int \kappa_D \sigma_\theta |v|^2 - \int \int \sigma_\theta \kappa_D |v|^2
\]

\[
= 2 \Re \int \int \sigma_\theta D^2 v \cdot \bar{v} + 2 \int \int \sigma_\theta |Dv|^2 + \int \int \kappa_D |v|^2 - 2 \Re \int \int \sigma_\theta \kappa_D |Dv| \cdot \bar{v}
\]

\[
= - 2 \Re \int \int \sigma_\theta (\mu \partial_\mu v + \rho^{-1} (\nu v)_\nu + q \nu v + \xi v^2 \nu + \xi v) \bar{v}
\]

\[
+ 2 \int \int \sigma_\theta |Dv|^2 + \int \int \sigma_\theta \kappa_D |v|^2 - \int \int (\sigma_\theta \kappa_D - D (\sigma_\theta \kappa_D)) |v|^2.
\]

Moreover, in view of

\[
2 \Re \int \int \sigma_\theta \mu \partial_\mu v \cdot \bar{v} = - \int \int (\sigma_\theta \mu) |v|^2,
\]

\[
2 \Re \int \int \sigma_\theta \xi \partial_\nu v \cdot \bar{v} = - \int \int \sigma_\theta \xi |v|^2
\]

we have

\[
\int \int |\phi| |v|^2 = \int \int (\sigma_\theta \mu) |v|^2 + 2 \int \int \sigma_\theta \rho v^2 |v|^2 + \int \int \sigma_\theta \xi |v|^2 - 2 \int \int \sigma_\theta (q + \xi) |v|^2
\]

\[
+ 2 \int \int \sigma_\theta |Dv|^2 + \int \int (\sigma_\theta \kappa_D - \sigma_\theta \kappa_D + D (\sigma_\theta \kappa_D)) |v|^2.
\]

Hence, we get

\[
\int_{r_1}^{r_1 + 1} \int_{\sigma_\theta}^{\sigma_{\theta + 1}} Fdr = \int_{r_1}^{\sigma_{\theta + 1}} Fdr
\]

\[
= \frac{1}{2} \int \int (\sigma_\theta - (\sigma_\theta \mu)_\mu - \sigma_\theta \kappa_D - D (\sigma_\theta \kappa_D)
\]

\[
+ \sigma_\theta (- \xi v + 4q + 2 \xi + 2 \eta + \kappa_D^2) |v|^2 - 2 \int \int \sigma_\theta \rho v^2 |v|^2
\]

(73)
\[ \leq \text{const.} \int |v|^2, \]

where we used that \( \sigma, \delta \sigma, \delta r, \mu, \mu r, \kappa, \kappa r, \xi, \xi r, \xi q \) and \( \xi \) are bounded. Consequently, from (3.6) we obtain the estimate

\[ \int_0^\infty \int_{S^1} |v(r, \theta)|^2 r d\theta dr \geq C \int_0^\infty \frac{dr}{r^2} \] \( (R \geq r) \).

But because the measure of \( M \) is \( \rho d\theta d\theta \) and \( v = \rho^{r/2} K u \), this corresponds to

\[ \int_{[r \leq \rho, \rho \leq 1]} |u(r, \theta)|^2 dM \geq C \int_0^\infty \frac{dr}{\rho (r)^2} - C' \]

which proves Theorem 1.1.

**Appendix**

We are stating Masuda's original theorem here, starting with his assumptions.

**Assumption 7.** There exists a locally integrable function \( \mu(r) \) such that for any \( x \in D \) and \( y \in \Omega(D(x)) \) satisfying \( \|y\|^2 + (A, x, x) > 0 \), we have

\[-2 \text{Re} \left( (B_1 + B_2) y, y + (A, x, x) \right) - 2 \text{Re} \left( (A, x, y) \right) \geq -\mu \cdot (\|y\|^2 + (A, x, x)) \]

**Assumption 8.** One can find a positive continuous function \( \gamma(r) \) satisfying

\[-2 \text{Re} \left( (B_1 + B_2) y, y - \text{Re} \left( [(B_1 + B_2) y] x, x \right) \right) \geq \gamma \cdot (r)|y|^2 \]

**Theorem A.** (K. Masuda). Under Assumptions 3-6, 7' and 8', every solution \( v \neq 0 \) of (2.1) satisfies either

\[ \|\dot{v} + B_1 v\|^2 + (A, (r) v (r), v (r)) \geq C \exp \left( -\int_{r_1}^r \mu(s) ds \right) \]

or

\[ \|v(r)\|^2 \geq C \exp \left( -\int_{r_1}^r \exp \left( -\int_{r_1}^s \beta(t) dt \right) ds \right) + C' \int_{r_1}^r \left( \int_{r_1}^s \beta(t) dt \right) (2 \gamma(r) - \beta(r)) \frac{dr}{r} ds \]

To prove this, he introduced \( G (r) := \|\dot{v} + B_1 v\|^2 + (A, v, v) \). By showing \( G (r) \geq 0 \) for large \( r \), he proved the second inequality unless there exists an \( r_2 \) such that \( G (r_2) > 0 \). Our Theorem 2.1 indicates that the second alternative does not occur under an additional condition which is natural from the viewpoint of some applications.

**References**


Growth Estimate of Generalized Eigenfunctions of $\Delta$ on Two-Dimensional Manifolds


